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Construction of Matrix in Multiplicative Matrix form

BAKSHI OM PRAKASH SINHA[1] and NARENDRA PRASAD[2]

[1]Department of Physics, Ramgarh College, Ramgarh Cantt (Jharkhand) (India)
[2]Department of Mathematics, Ramgarh College, Ramgarh Cantt (Jharkhand) (India)
Corresponding Author Email:- bakshi.op@gmail.com
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Abstract

In this chapter we investigate n x n matrix M which is in multiplicative matrix form. This is a quite new concept. A theorem is given to construct an infinite number of matrices M in multiplicative matrix form of different order even if matrix A has only two entries a & b taken alternatively in its first row or first column with examples.

Key words : Multiplicative matrix form[1], Circulant matrix[2], Triangular matrix[3], Upper triangular matrix[4], Lower triangular matrix[5] & Hadamart product[6].

NOTATIONS

J_n : nxn matrix having all entries 1

J^{mxn} : mxn matrix having all entries 1

$-J^{mxn}$: mxn matrix having all entries -1

M^T : transpose of matrix M

Definition

1) Multiplicative Matrix Form

An nxn matrix M is said to be in multiplicative matrix form if

$M(a_1, a_2, \ldots\ldots\ldots, a_r)\, M(b_1, b_2, \ldots\ldots\ldots, b_r) = M(c_1, c_2, \ldots\ldots\ldots, c_r)$

where, $C_i = \sum_{jk} c^i{}_j a_j b_k$ is a bilinear function of a's and b's.

Example Let,

$$M(a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \quad (1)$$

$$\text{and } M(a_1, a_2, a_3) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

be two 3x3 matrices. we consider,

$$M(a_1, a_2, a_3)\, M(a_1, a_2, a_3) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}, \text{ by (1) \& 2)}$$

$$= \begin{pmatrix} a_1 b_1 & 0 & 0 \\ 0 & a_2b_2 & a_3b_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & c_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= M(c_1, c_2, c_3)$$

where $c_1 = a_1b_1$, $c_2 = a_2b_2$, and $c_3 = a_3b_3$

2) ***Circulant Matrix :***

A matrix M is called a circulant matrix if the next entry of rows or columns starts from the entry where it ends in previous rows or columns and the arrangements of entries in each row or column are order as it is taken in first row or column. e.g

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

is a Circulant matrix

3) ***Triangular Matrix***

An nxn matrix $A = [a_{ij}]$ is called a triangular matrix if $a_{ij} = 0$, either $i<j$ or $i>j$.

4) ***Upper triangular matrix***

An nxn matrix $A = [a_{ij}]$ is called an Upper Triangular Matrix if $a_{ij} = 0$, $i>j$.

Example

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 5 & 1 & 2 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

is an upper triangular matrix.

5) ***Lower Triangular Matrix***

An nxn matrix $A = [a_{ij}]$ is called an Lower Triangular Matrix if $a_{ij} = 0$, if $i<j$.

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 2 \end{pmatrix}$$

is a lower triangular matrix.

6) ***Hadamard Product of two matrices :*** Let A and B are two matrices of the same type. Then the matrix obtained by the product of corresponding entries of matrices. A and B is called Hadamard product of two matrices A and B. It is denoted by $A \bullet B$.

Let,

$$A = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$$

then,

$$A \bullet B = \begin{pmatrix} m_1 a_1 & m_2 a_2 & m_3 a_3 \\ m_4 a_4 & m_5 a_5 & m_6 a_6 \end{pmatrix}$$

Theorem

If an nxn matrix A is in multiplicative matrix form with two symbols a and b in rows or columns taken alternatively, then the symmetrical matrix

$$M = A \bullet J_n$$

is in multiplicative matrix form of order $n=m(r+2s)$ where r and s are numbers of a's and b's in matrix A and $J_n = [\, R \;\; R_1 \,\ldots\ldots\ldots\, R \;\; R_1 \,]^T$, here $R = [J_m \;\; J_{mx2m} \;\ldots\ldots\; J_m \;\; J_{mx2m}]$, $R_1 = [J^T_{mx2m} \;\; J_{2m} \;\ldots.\; J^T_{mx2m} \;\; J_{2m}\,]$ and $\bullet$ stands for Hadamard product.

Proof :- Let

$$A = \begin{pmatrix} a & b & a & \ldots\ldots & b \\ b & a & b & \ldots\ldots & a \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ b & a & b & \ldots\ldots & a \end{pmatrix} \qquad (1)$$

be in nxn matrix in a multiplicative matrix form.

Then matrix,

$$M = A \bullet J_{m(r+2s)} = M\,(a,b) \qquad (2)$$

$$= \begin{pmatrix} a & b & \ldots\ldots & a & b \\ b & a & \ldots\ldots & b & a \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ b & a & \ldots\ldots & b & a \end{pmatrix} \bullet \begin{pmatrix} J_m & J_{mx2m} & \ldots. & J_{mx2m} \\ J^T_{mx2m} & J_{2m} & \ldots. & J_{2m} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ J^T_{mx2m} & J_{2m} & \ldots. & J_{2m} \end{pmatrix}$$

$$= \begin{pmatrix} aJ_m & bJ_{mx2m} \ldots\ldots & bJ_{mx2m} \\ bJ^T_{mx2m} & aJ_{2m} \;\ldots\ldots & aJ_{2m} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ bJ^T_{mx2m} & aJ_{2m} \;\ldots\ldots & aJ_{2m} \end{pmatrix}$$

be a matrix of order m(r+2s). We show that the matrix M is in multiplicative matrix form.

We consider

M (a_1, b_1) M(a_2, b_2)

$$= \begin{pmatrix} a_1J_m & b_1J_{mx2m} & \ldots & b_1J_{mx2m} \\ b_1J^T_{mx2m} & a_1J_{2m} & \ldots & a_1J_{2m} \\ \ldots & & & \\ \ldots & & & \\ b_1J^T_{mx2m} & a_1J_{2m} & \ldots & a_1J_{2m} \end{pmatrix} \bullet \begin{pmatrix} a_2J_m & b_2J_{mx2m} & b_2J_{mx2m} \\ a_2J^T_{mx2m} & a_2J_{2m} & a_2J_{2m} \\ \ldots & & \\ \ldots & & \\ a_2J^T_{mx2m} & a_2J_{2m}\ldots & a_2J_{2m} \end{pmatrix}$$

$$= \begin{pmatrix} cJ_m & dJ_{mx2m} & \ldots & dJ_{mx2m} \\ dJ^T_{mx2m} & cJ_{2m} & \ldots & cJ_{2m} \\ \ldots & & & \\ \ldots & & & \\ dJ^T_{mx2m} & cJ_{2m} & \ldots & cJ_{2m} \end{pmatrix} = M(c, d)$$

which shows that matrix M is in multiplicative matrix form

where $C = m\,(ra_1a_2 + 2sb_1b_2)$

and $d = m(ra_1b_2 + 2sb_1a_2)$ and they are bilinear function of a's and b's.

Remark :

1. Order nxn of matrix M depends upon the number of a's and b's in matrix A and m in J_n.
2. R contains as many as blocks of J_m and J_{mx2m} as A has number of entries in first row or first column.
3. We can construct matrix M with more than 2 symbols suitably placed in its rows and columns.

Example - 1 : Let,

$$A = \begin{pmatrix} a & b & a \\ b & a & b \\ a & b & a \end{pmatrix} \qquad (1)$$

be a 3x3 matrix in multiplicative matrix form with two a's and one b's. Here we take m = 2 and we have r = 2, & s = 1 from (i), We construct a 8x8 matrix M of order 2(2+2.1)=8. The block matrix J_8 will be of the form as

$$J_8 = [\, R \;\; R_1 \;\; R \,]^T \qquad (2)$$

Where $R = [J_2 \;\; J_{2x4} \;\; J^2]$ and (3)

$$R_1 = [J^T_{2x4} \;\; J_4 \;\; J^T_{2x4}] \qquad (4)$$

Therefore, the matrix M is constructed as

$M = A \bullet J_8 = A\bullet [\, R \;\; R_1 \;\; R \,]^T$

$$= \begin{pmatrix} a & b & c \\ b & a & b \\ a & b & c \end{pmatrix} \bullet \begin{pmatrix} J_2 & J_{2x4} & J_2 \\ J^T_{2x4} & J_{4x4} & J^T_{2x4} \\ J_2 & J_{2x4} & J_2 \end{pmatrix},$$

by (3) and (4)

$$= \begin{pmatrix} aJ_2 & bJ_{2x4} & aJ_2 \\ bJ^T_{2x4} & aJ_{4x4} & bJ^T_{2x4} \\ aJ_2 & bJ_{2x4} & aJ_2 \end{pmatrix}$$

$$= \begin{pmatrix} a & a & b & b & b & b & a & a \\ a & a & b & b & b & b & a & a \\ b & b & a & a & a & a & b & b \\ b & b & a & a & a & a & b & b \\ b & b & a & a & a & a & b & b \\ b & b & a & a & a & a & b & b \\ a & a & b & b & b & b & a & a \\ a & a & b & b & b & b & a & a \end{pmatrix} = M(a, b) \qquad (5)$$

We show that the matrix M is in multiplicative matrix form

We consider, $M(a_1, b_1)\, M(a_2, b_2)$

$$= \begin{pmatrix} a_1 & a_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 \\ a_1 & a_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 \\ b_1 & b_1 & a_1 & a_1 & a_1 & a_1 & b_1 & b_1 \\ b_1 & b_1 & a_1 & a_1 & a_1 & a_1 & b_1 & b_1 \\ b_1 & b_1 & a_1 & a_1 & a_1 & a_1 & b_1 & b_1 \\ b_1 & b_1 & a_1 & a_1 & a_1 & a_1 & b_1 & b_1 \\ a_1 & a_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 \\ a_1 & a_1 & b_1 & b_1 & b_1 & b_1 & a_1 & a_1 \end{pmatrix}$$

$$\begin{pmatrix} a_2 & a_2 & b_2 & b_2 & b_2 & b_2 & a_2 & a_2 \\ a_2 & a_2 & b_2 & b_2 & b_2 & b_2 & a_2 & a_2 \\ b_2 & b_2 & a_2 & a_2 & a_2 & a_2 & b_2 & b_2 \\ b_2 & b_2 & a_2 & a_2 & a_2 & a_2 & b_2 & b_2 \\ b_2 & b_2 & a_2 & a_2 & a_2 & a_2 & b_2 & b_2 \\ b_2 & b_2 & a_2 & a_2 & a_2 & a_2 & b_2 & b_2 \\ a_2 & a_2 & b_2 & b_2 & b_2 & b_2 & a_2 & a_2 \\ a_2 & a_2 & b_2 & b_2 & b_2 & b_2 & a_2 & a_2 \end{pmatrix}$$

$$= \begin{pmatrix} c & c & d & d & d & d & c & c \\ c & c & d & d & d & d & c & c \\ d & d & c & c & c & c & d & d \\ d & d & c & c & c & c & d & d \\ d & d & c & c & c & c & d & d \\ d & d & c & c & c & c & d & d \\ c & c & d & d & d & d & c & c \\ c & c & d & d & d & d & c & c \end{pmatrix}$$

Where $c = 2(2a_1a_2 + 2b_1b_2)$

and $d = 2(2a_1b_2 + 2b_1a_2)$

$$= \begin{pmatrix} cJ_2 & dJ_{2x4} & cJ_2 \\ dJ^T_{2x4} & cJ_{4x4} & dJ^T_{2x4} \\ cJ_2 & dJ_{2x4} & cJ_2 \end{pmatrix} = M(c, d).$$

$\therefore$ $M(a_1, b_1)\, M(a_2, b_2) = M(c, d)$.

$\therefore$ Matrix M is in multiplicative matrix form.

Example - 2 : Let,

$$A = \begin{pmatrix} a & b & a \\ b & a & b \\ a & b & a \end{pmatrix} \quad (1)$$

matrix in a multiplicative matrix form, having two a's and one b's.

Here we take m=3 and we have 2 a's and one b's from (i). We construct a matrix M of order 3(2+2.1)=12. The block matrix J_{12} will of the form as,

$$J_{12} = [\,R \;\; R_1 \;\; R]^T \quad (2)$$

where $R = [J_3 \;\; J_{3x6} \;\; J_3]$ and (3)

$$R_1 = [J^T_{3x6} \;\; J_6 \;\; J^T_{3x6}] \quad (4)$$

Therefore, the matrix M is constructed as

$M = A \bullet J_{12} = A \bullet [\,R \;\; R_1 \;\; R\,]^T$

$$= \begin{pmatrix} a & b & a \\ b & a & b \\ a & b & c \end{pmatrix} \bullet \begin{pmatrix} J_3 & J_{3x6} & J_3 \\ J^T_{3x6} & J_{6x6} & J^T_{3x6} \\ J_3 & J_{3x6} & J_3 \end{pmatrix} \quad \text{by (3) and (4)}$$

$$= \begin{pmatrix} aJ_3 & bJ_{3x6} & aJ_3 \\ bJ^T_{3x6} & aJ_{6x6} & bJ^T_{3x6} \\ aJ_3 & bJ_{3x6} & aJ_3 \end{pmatrix}$$

$$= \begin{pmatrix} a & a & a & b & b & b & b & b & b & a & a & a \\ a & a & a & b & b & b & b & b & b & a & a & a \\ a & a & a & b & b & b & b & b & b & a & a & a \\ b & b & b & a & a & a & a & a & a & b & b & b \\ b & b & b & a & a & a & a & a & a & b & b & b \\ b & b & b & a & a & a & a & a & a & b & b & b \\ b & b & b & a & a & a & a & a & a & b & b & b \\ b & b & b & a & a & a & a & a & a & b & b & b \\ b & b & b & a & a & a & a & a & a & b & b & b \\ a & a & a & b & b & b & b & b & b & a & a & a \\ a & a & a & b & b & b & b & b & b & a & a & a \\ a & a & a & b & b & b & b & b & b & a & a & a \end{pmatrix}_{12x12}$$

$$= M(a, b) \quad (5)$$

we show that matrix M is in multiplicative form i.e.

We consider,

$M(a_1,b_1)\; M(a_2,b_2)$

$$= \begin{pmatrix} a_1J_3 & b_1J_{3x6} & a_1J_3 \\ b_1J^T_{3x6} & a_1J_{6x6} & b_1J^T_{3x6} \\ a_1J_3 & b_1J_{3x6} & a_1J_3 \end{pmatrix} \begin{pmatrix} a_2J_3 & b_2J_{3x6} & a_2J_3 \\ b_2J^T_{3x6} & a_2J_{6x6} & b_2J^T_{3x6} \\ a_2J_3 & b_2J_{3x6} & a_2J_3 \end{pmatrix}$$

$$= \begin{pmatrix} cJ_3 & dJ_{3x6} & cJ_3 \\ dJ^T_{3x6} & cJ_{6x6} & dJ^T_{3x6} \\ cJ_3 & dJ_{3x6} & cJ_3 \end{pmatrix} = M(c, d)$$

Where $\quad c = 3(2a_1a_2 + 2b_1b_2) = 6\,(a_1a_2 + b_1b_2)$

and $\quad d = 3(2a_1b_2 + 2b_1a_2) = 6\,(a_1b_2 + b_1a_2)$.

$= M\,(c, d)$.

$\therefore\; M(a_1, b_1)\, M(a_2, b_2) = M\,(c, d)$.

which shows that matrix M is in the multiplicative matrix form.

Example - 3 : Let A be a matrix of even order are in multiplicative form.

$$A = \begin{pmatrix} a & b & a & b \\ b & a & b & a \\ a & b & a & b \\ b & a & b & a \end{pmatrix} \qquad (1)$$

be a 4x4 matrix in multiplicative matrix form having two a's and two b's. Here, we take m=2 and we have two a's and two b's from (1). We construct a matrix M of order 2(2+2.2)=12. The block matrix J_{12} will of the form as,

$$J_{12} = [\,R\;\; R_1\;\; R\;\; R_1]^T \qquad (2)$$

where $$R = [\,J_2\;\; J_{2x4}\;\; J_2\; J_{2x4}] \text{ and} \qquad (3)$$

$$R_1 = [\,J^T_{2x4}\;\; J_4\;\; J^T_{2x4}\; J_4] \qquad (4)$$

Therefore, the matrix M is constructed as

$$M = A \bullet J_{12} = A \bullet [\,R\;\; R_1\;\; R\;\;\; R_1]^T$$

$$= \begin{pmatrix} a & b & a & b \\ b & a & b & a \\ a & b & c & b \\ b & a & b & a \end{pmatrix} \bullet \begin{pmatrix} J_2 & J_{2x4} & J_2 & J_{2x4} \\ J^T_{2x4} & J_4 & J^T_{2x4} & J_4 \\ J_2 & J_{2x4} & J_2 & J_{2x4} \\ J^T_{2x4} & J_4 & J^T_{2x4} & J_4 \end{pmatrix}$$

by (3) and (4)

$$= \begin{pmatrix} aJ_2 & bJ_{2x4} & aJ_2 & bJ_{2x4} \\ bJ^T_{2x4} & aJ_4 & bJ^T_{2x4} & aJ_4 \\ aJ_2 & bJ_{2x4} & aJ_2 & bJ_{2x4} \\ bJ^T_{2x4} & aJ_4 & bJ^T_{2x4} & aJ_4 \end{pmatrix}$$

$$= \begin{pmatrix} a & a & b & b & b & b & a & a & b & b & b & b \\ a & a & b & b & b & b & a & a & b & b & b & b \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \\ a & a & b & b & b & b & a & a & b & b & b & b \\ a & a & b & b & b & b & a & a & b & b & b & b \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \\ b & b & a & a & a & a & b & b & a & a & a & a \end{pmatrix}_{12x12}$$

Table - 1. Construction of Matrix 'M'
Where o(A) and o(M) denote the order of matrices A and M respectively
r = numbers of a's & s = numbers of b's

o(A)	**r**	**s**	**m**	**o(M) =m(r+2s)**	**c**	**d**
2x2	1	1	2	6	$2(a_1a_2+2b_1b_2)$	$2(a_1b_2+2b_1a_2)$
2x2	1	1	3	9	$3(a_1a_2+2b_1b_2)$	$3(a_1b_2+2b_1a_2)$
2x2	1	1	4	12	$4(a_1a_2+2b_1b_2)$	$4(a_1b_2+2b_1a_2)$
..	..	..	..	..	..	..
3x3	2	1	2	8	$4(a_1a_2+b_1b_2)$	$4(a_1b_2+b_1a_2)$
3x3	2	1	3	12	$6(a_1a_2+b_1b_2)$	$6(a_1b_2+b_1a_2)$
3x3	2	1	4	16	$8(a_1a_2+b_1b_2)$	$8(a_1b_2+b_1a_2)$
..	..	..	..	..	..	..
4x4	2	2	2	12	$4(a_1a_2+2b_1b_2)$	$4(a_1b_2+2b_1a_2)$
4x4	2	2	3	18	$6(a_1a_2+2b_1b_2)$	$6(a_1b_2+2b_1a_2)$
4x4	2	2	4	24	$8(a_1a_2+2b_1b_2)$	$8(a_1b_2+2b_1a_2)$
..	..	..	..	..	..	..

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Suggestion / Further scopes :-

a) Order nxn of matrix M depends upon the number of a's and b's in matrix A and m in J_n.
b) 'R' contains as many as blocks of J_m and J_{mx2m} as A has number of entries in first row or first column.
c) We can construct matrix M with more than 2 symbols suitably placed in its rows and columns.
d) It preserves the matrix like Circulant matrix, Upper triangular matrix, lower triangular matrix & orthogonal matrix and further research work may be done in this field
e) By assigning any value of 'n' any metrics form can be obtained and the process can be made very easy to solve the difficult & rig ours problem.

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