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On W_0- Curvature Tensor of Generalized Sasakian-Space-Forms

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Abstract

The object of the present paper is to study generalized Sasakian-space-forms satisfying certain curvature conditions on W_0- curvature tensor. In this paper, we study W_0- semisymmetric, W_0-flat, $\xi-W_0-$ flat, generalized Sasakian-space-forms satisfying $A.S=0$, $A.R=0$, $A.\tilde{C}=0$. Also $\Phi-W_0-$ flat generalized Sasakian-space-form have been studied.

Key words : Generalized Sasakian-space form, W_0- curvature tensor, Concircular curvature tensor, Ricci tensor, $\Phi-W_0-$ flat, $\eta-$Einstien Manifold, scalar curvature

2000 Subject Classification Code - 53C25, 53D15.

1 Introduction

In 1971, G.P. Pokhariyal and R.S. Mishra[5] introduced and studied a new curvature tensor W^* on Riemannian manifold and studied its relativistic significance. In 2011, M.M. Tripathi and P. Gupta[7] introduced and studied $\tau-$ curvature tensor in semi -Riemannian manifolds. They give the properties and some identities of $\tau-$ curvature tensor. They define W_0- curvature tensor of type $(0,4)$ for $(2n+1)-$dimensional Riemannian manifold, as

$$W_0(X,Y,Z,U)=R(X,Y,Z,U)-\frac{1}{2n}\{S(Y,Z)g(X,U)-g(X,Z)S(Y,U)\} \qquad (1.1)$$

where R and S denote the Riemannian curvature tensor of type $(0,4)$ defined by $'R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type $(0,2)$ respectively. The curvature tensor defined by (1.1) is known as W_0- curvature tensor. A manifold whose W_0- curvature tensor vanishes at every point of the manifold is called W_0- flat manifold. They also define $\tau-$conservative semi $\_$ Riemannian manifolds and give necessary and sufficient condition for semi $\_$ Riemannian manifolds to be $\tau-$ conservative.

U.C. De and A. Sarkar[12] studied generalized Sasakian-space-forms with vanishing quasi-conformal curvature tensor and quasi-conformal flat generalized Sasakian-space-forms, Ricci-symmetric and Ricci semisymmetric generalized Sasakian-space-forms[5].

In[2], C. Özgür and M.M. Tripathi obtained the necessary and sufficient conditions for curvatures of P-Sasakian manifolds satisfying certain conditions.

P. Alegre, D. Blair and A. Carriazo[9] introduced and studied generalized Sasakian-space-forms. These space-forms are defined as follows:

Given an almost contact metric manifold $M(\Phi,\xi,\eta,g)$, we say that M is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\Phi Z)\Phi Y - g(Y,\Phi Z)\Phi X + 2g(X,\Phi Y)\Phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

for any vector fields X,Y,Z on M. In such a case, we denote the manifold as $M(f_1, f_2, f_3)$. This kind of manifold appears as a generalization of the well known Sasakian-space-form, which can be obtained as a particular case of generalized Sasakian-space-forms by taking $f_1 = \frac{c+3}{4},\ f_2 = f_3 = \frac{c-1}{4}$. It is known that any three-dimensional $(\alpha,\beta)-$ trans Sasakian manifold with α,β depending on ξ is a generalized Sasakian-space-form[8]. G. Zhen and etar have given results in[6] about $\xi-$ conformally flat contact metric manifolds. In [3] C. Özgür studied $\Phi\_$ conformally flat LP- Sasakian manifolds. M.M. Tripathi and P. Gupta give results in[7] about $\tau-$ curvature tensor on a semi-Reimannian manifold. Sreenivasa. G.T. Venkatesha and Bagewadi C.S. [10] have some results on $(LCS)_{2n+1}-$Manifolds. S. K. Yadav, P.K. Dwivedi and D. Suthar[11] studied $(LCS)_{2n+1}-$ Manifolds satisfying certain conditions on the concircular curvature tensor. Motivated by above studies, we study the W_0- curvature tensor of generalized Sasakian-space-forms. The present paper is organised as follows:

In section 2, some preliminary results are recalled. In section 3, we study W_0- semisymmetric generalized Sasakian-space-forms. Section 4 deals with $\xi-W_0$ flat generalized Sasakian-space-forms. Section 5, we study generalized Sasakian-space-form satisfying $A.S = 0$. In section 6, W_0- flat generalized Sasakian-space-forms are studied. Section 7 deals with $\Phi-W_0$ generalized Sasakian-space-form. Section 8, we study generalized Sasakian-space-form satisfying $A.R = 0.$ Last section[9] contains, generalized Sasakian-space-form satisfying $A.\tilde{C} = 0.$

2 Preliminaries :

An odd $-$ dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function Φ, a 1-form $\eta,$ associated vector field ξ and the Riemannian metric g satisfying

$$\Phi^2(X) = -X + \eta(X)\xi, \Phi(\xi) = 0 \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\Phi X) = 0 \tag{2.2}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for arbitary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifod[4], and the structure (Φ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of (2.1), (2.2) and (2.3), we have

$$g(\Phi X, Y) = -g(X, \Phi Y), g(\Phi X, X) = 0 \tag{2.4}$$

$$\nabla_X \eta(Y) = g(\nabla_X \xi, Y) \tag{2.5}$$

Again we know[9] that in a $(2n+1)-$ dimensional generalized-Sasakian-space-form, we have

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\Phi Z)\Phi Y - g(Y,\Phi Z)\Phi X + 2g(X,\Phi Y)\Phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.6}$$

for any vector field X, Y, Z on $M^{2n+1},$ where R denotes the curvature tensor of M^{2n+1} and f_1, f_2, f_3 are smooth functions on the manifold.

The Ricci tensor S and the scalar curvature r of the manifold of dimension $(2n+1)$ are respectively, given by

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \tag{2.9}$$

Also for a generalized Sasakian-space-forms, we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{2.10}$$

$$R(\xi, X)Y = -R(X,\xi)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \tag{2.11}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X) \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi \tag{2.14}$$

where Q is the Ricci Operator, i.e.

$$g(QX,Y) = S(X,Y) \tag{2.15}$$

For a $(2n+1)-$ dimensional $(n>1)$ Almost Contact Metric, the $W_0 -$ curvature tensor A is given by

$$A(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - g(X,Z)QY\} \tag{2.16}$$

the $W_0 -$ curvature tensor A in a generalized Sasakian-space-form satisfies

$$A(X,Y)\xi = -(f_1 - f_3)\eta(X)Y + \frac{1}{2n}\{(2nf_1 + 3f_2 - f_3)\eta(X)Y - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y)\xi \tag{2.17}$$

$$A(\xi,Y)\xi = (f_1 - f_3)\{\eta(Y)\xi - Y\}$$

$$-\frac{1}{2n}\{2n(f_1 - f_3)\eta(Y)\xi - (2nf_1 + 3f_2 - f_3)Y + (3f_2 + (2n-1)\eta(Y)\xi \tag{2.18}$$

$$A(X,\xi)\xi = 0 \tag{2.19}$$

$$A(\xi,X)Y = \frac{1}{2n}((1-2n)f_3 - 3f_2)(g(X,Y)\xi - \eta(Y)X) \tag{2.20}$$

Given an $(2n+1)-$ dimensional Riemannian manifold (M,g), the Concircular curvature tensor $\tilde{C}$ is given by

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{2.21}$$

$$\tilde{C}(\xi,X)Y = [f_1 - f_3 - \frac{r}{2n(2n+1)}]\{g(X,Y)\xi - \eta(Y)X\} \tag{2.22}$$

and

$$\eta(\tilde{C}(X,Y)Z) = [f_1 - f_3 - \frac{r}{2n(2n+1)}]\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{2.23}$$

3 $W_0 -$ Semisymmetric Generalized Sasakian-Space-Forms :

Definition 1. A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-Space-Form is said to be $W_0 -$ semisymmetric if it satisfies $R.A = 0$, where R is the Riemannian curvature tensor, A is the $W_0 -$ curvature tensor of the space forms.

Theorem 1. A $(2n+1)-$ *dimensional* $(n>1)$ *generalized Sasakian-space-form is* W_0- *semisymmetric if and only if* $f_1 = f_3$.

Proof. Let us suppose that the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is W_0- semisymmetric, then we have

$$R(\xi,U).A(X,Y)\xi = 0 \tag{3.1}$$

The above equation can be written as

$$R(\xi,U)A(X,Y)\xi - A(R(\xi,U)X,Y)\xi - A(X,R(\xi,U)Y)\xi - A(X,Y)R(\xi,U)\xi = 0 \tag{3.2}$$

In view of (2.2), (2.10) & (2.11) the above equation reduces to

$$\begin{aligned}(f_1-f_3)\{&g(U,A(X,Y)\xi)\xi - \eta(A(X,Y)\xi)U - g(U,X)A(\xi,Y)\xi \\ &+\eta(X)A(U,Y)\xi - g(U,Y)A(X,\xi)\xi + A(X,U)\eta(Y)\xi \\ &- A(X,Y)\eta(U)\xi + A(X,Y)U\} = 0\end{aligned} \tag{3.3}$$

In view of (2.15), (2.16) & (2.17) and taking the Inner Product of above equation with ξ, we get

$$(f_1-f_3)\{g(U,A(X,Y)\xi) + g(A(X,Y)U,\xi)\} = 0 \tag{3.4}$$

$$(f_1-f_3)\{g(U,A(X,Y)\xi) + \eta(A(X,Y)U)\} = 0 \tag{3.5}$$

This implies either $f_1 = f_3$ or

$$g(U,A(X,Y)\xi) + \eta(A(X,Y)U) = 0 \tag{3.6}$$

which by using (2.14) and (2.15) gives

$$\eta(Y)g(X,U) - \eta(X)g(U,Y) = 0 \tag{3.7}$$

which is not possible in generalized sasakian-space-form. Conversely, if $f_1 = f_3$, then from (2.11), $R(\xi,U) = 0$. Then obviously $R.A = 0$ is satisfies. This completes the proof. □

4 $\xi - W_0-$ *Flat Generalized Sasakian-Space-Forms :*

Definition 2. A $(2n+1)-$ *dimensional* $(n>1)$ *generalized Sasakian-space-form is said to be* W_0- *flat*[6] *if* $A(X,Y)\xi = 0$ *for all* $X,Y \in TM$.

Theorem 2. A $(2n+1)-$ *dimensional* $(n>1)$ *generalized Sasakian-space-form is* $\xi - W_0-$ *flat if and only if it is an Einstein Manifold.*

Proof. Let us consider that a generalized Sasakian-space-form is $\xi - W_0-$ flat, i.e. $A(X,Y)\xi = 0$. Then from (2.16), we have

$$R(X,Y)\xi = \frac{1}{2n}\{S(Y,\xi)X - g(X,\xi)QY\} \tag{4.1}$$

$$R(X,Y)\xi = \frac{1}{2n}\{S(Y,\xi)X - \eta(X)QY\} \tag{4.2}$$

By using (2.10) & (2.12) above equation becomes

$$(f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} = \frac{1}{2n}\{2n(f_1 - f_3)\eta(Y)X - \eta(X)QY\} \tag{4.3}$$

On solving, we get

$$QY = 2n(f_1 - f_3)Y \tag{4.4}$$

Now, taking the inner product of the above equation with U, we get

$$S(Y,U) = 2n(f_1 - f_3)g(Y,U) \tag{4.5}$$

which implies generalized Sasakian-space-form is an Einstein Manifold. Conversely, suppose that (4.5) is satisfied. Then from (4.1) & (4.4), we get

$$A(X,Y)\xi = 0$$

This completes the proof. □

5 *Generalized Sasakian-Space-Form Satisfying* $A.S = 0$

Theorem 3. A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ *satisfies the condition* $A(\xi, X).S = 0$ *if and only if either* $M^{2n+1}(f_1, f_2, f_3)$ *has* $3f_2 = (1-2n)f_3$ *or is an Einstein Manifold.*

Proof. The condition $A(\xi, X).S = 0$ implies that

$$S(A(\xi, X)Y, Z) + S(Y, A(\xi, X)Z) = 0$$

for any vector fields X, Y, Z on M. Substituting (2.18) in above equation, we obtain

$$\frac{1}{2n}((1-2n)f_3 - 3f_2)\{g(X,Y)S(Z,\xi) - \eta(Y)S(X,Z) + g(X,Z)S(Y,\xi) - \eta(Z)S(X,Y)\} = 0 \tag{5.1}$$

For $Z = \xi$, the last equation is equivalent to

$$\frac{1}{2n}((1-2n)f_3 - 3f_2)\{g(X,Y)S(\xi,\xi) - \eta(Y)S(X,\xi) + g(X,\xi)S(Y,\xi) - \eta(\xi)S(X,Y)\} = 0 \tag{5.2}$$

Using (2.12), we get

$$((1-2n)f_3 - 3f_2)(S(X,Y) - 2n(f_1 - f_3)g(X,Y)) = 0 \tag{5.3}$$

which proves our assertion. □

6 W_0- *flat Generalized Sasakian-space-forms :*

Theorem 4. A $(2n+1)-$ *dimensional* $(n>1)$ *generalized Sasakian-space-form is* W_0- *flat if and only if* $f_1 = \frac{3f_2}{(1-2n)} = f_3$.

Proof. For a $(2n+1)-$ dimensional W_0- flat generalized Sasakian-space-form, we have from (2.16)

$$R(X,Y)Z = \frac{1}{2n}\{S(Y,Z)X - g(X,Z)QY\} \tag{6.1}$$

In view of (2.7) & (2.8), the above equation takes the form

$$R(X,Y)Z = \frac{1}{2n}\{(2nf_1 + 3f_2 - f_3)(g(Y,Z)X - g(X,Z)Y)$$
$$+(3f_2 + (2n-1)f_3)(g(X,Z)\eta(Y)\xi - \eta(X)\eta(Z)Y\} \tag{6.2}$$

By virtue of (2.6) the above equation reduces to

$$f_1\{g(Y,Z)X - g(X,Z)Y\}$$
$$+ f_2\{g(X,\Phi Z)\Phi Y - g(Y,\Phi Z)\Phi X + 2g(X,\Phi Y)\Phi Z\}$$
$$+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$
$$= \frac{1}{2n}\{(2nf_1 + 3f_2 - f_3)(g(Y,Z)X - g(X,Z)Y)$$
$$+(3f_2 + (2n-1)f_3)(g(X,Z)\eta(Y)\xi - \eta(X)\eta(Z)Y\} \tag{6.3}$$

Now, replacing Z by ΦZ in the above equation, we obtain

$$f_2\{g(X,\Phi^2 Z)\Phi Y - g(Y,\Phi^2 Z)\Phi X + 2g(X,\Phi Y)\Phi^2 Z\}$$
$$+ f_3\{g(X,\Phi Z)\eta(Y)\xi - g(Y,\Phi Z)\eta(X)\xi\}$$
$$= \frac{1}{2n}\{(3f_2 - f_3)(g(Y,\Phi Z)X - g(X,\Phi Z)Y)$$
$$+(3f_2 + (2n-1)f_3)g(X,\Phi Z)\eta(Y)\xi\} \tag{6.4}$$

taking inner product of above equation with ξ, we get

$$f_2\{g(X,\Phi^2 Z)g(\Phi Y,\xi) - g(Y,\Phi^2 Z)g(\Phi X,\xi) + 2g(X,\Phi Y)g(\Phi^2 Z,\xi)\}$$
$$+ f_3\{g(X,\Phi Z)\eta(Y)g(\xi,\xi) - g(Y,\Phi Z)\eta(X)g(\xi,\xi)\}$$
$$= \frac{1}{2n}\{(3f_2 - f_3)[g(Y,\Phi Z)g(X,\xi) - g(X,\Phi Z)g(Y,\xi)]$$
$$+(3f_2 + (2n-1)f_3)g(X,\Phi Z)\eta(Y)g(\xi,\xi)\} \tag{6.5}$$

In view of (2.2) & (2.1), we obtain

$$(-3f_2 - (2n-1)f_3)g(Y,\Phi Z)\eta(X) = 0 \tag{6.6}$$

Putting $X = \xi$ in above equation, we get

$$(-3f_2-(2n-1)f_3)g(Y,\Phi Z)=0 \tag{6.7}$$

Since $g(Y,\Phi Z)\neq 0$, in general, we obtain

$$-3f_2-(2n-1)f_3=0 \tag{6.8}$$

This implies

$$f_3=\frac{3f_2}{1-2n} \tag{6.9}$$

$$f_1=\frac{3f_2}{1-2n}=f_3 \tag{6.10}$$

Conversely, suppose that $f_1=\frac{3f_2}{1-2n}=f_3$ satisfies in generalized Sasakian-space-form, and then we have

$$S(X,Y)=0 \tag{6.11}$$

$$QX=0 \tag{6.12}$$

Also, in view of (2.14), we have

$$A(X,Y,Z,U)={}'R(X,Y,Z,U) \tag{6.13}$$

where $A(X,Y,Z,U)=g(X,Y,Z,U)$ and ${}'R(X,Y,Z,U)=g(X,Y,Z,U)$. Putting $Y=Z=e_i$ in above equation and taking summation over $i, 1\leq i\leq 2n+1,$ we get

$$\sum_{i=1}^{2n+1}A(X,e_i,e_i,U)=\sum_{i=1}^{2n+1'}R(X,e_i,e_i,U)=S(X,U) \tag{6.14}$$

In view of (2.6) & (6.13), we have

$$\begin{aligned}A(X,Y,Z,U)={}&f_1\{g(Y,Z)g(X,U)-g(X,Z)g(Y,U)\}\\&+f_2\{g(X,\Phi Z)g(\Phi Y,U)-g(Y,\Phi Z)g(\Phi X,U)+2g(X,\Phi Y)g(\Phi Z,U)\}\\&+f_3\{\eta(X)\eta(Z)g(Y,U)-\eta(Y)\eta(Z)g(X,U)\\&+g(X,Z)\eta(Y)\eta(U)-g(Y,Z)\eta(X)\eta(U)\}\end{aligned} \tag{6.15}$$

Now, putting $Y=Z=e_i$ in above equation and taking summation over $i, 1\leq i\leq 2n+1,$ we get

$$\sum_{i=1}^{2n+1}A(X,e_i,e_i,U)=f_1\{2ng(X,U)\}+3f_2g(X,U)-f_3\{(2n+1)\eta(X)\eta(U)+g(X,U)\} \tag{6.16}$$

In view of $(6.11), (6.12)$ & (6.15), we have

$$2nf_1g(X,U)+3f_2g(\Phi X,\Phi U)-f_3\{(2n+1)\eta(X)\eta(U)+g(X,U)\}=0 \tag{6.17}$$

Putting $X=U=e_i$ in above equation and taking summation over $i, 1\leq i\leq 2n+1,$ we get $f_1=0$.

Then in view of (6.10), $f_2 = f_3 = 0.$ Therefore, we obtain from (2.6)

$$R(X,Y)Z = 0 \tag{6.18}$$

Hence in view of (6.11), (6.12) & (6.17), we have $A(X,Y)Z = 0.$ This completes the proof. □

7 $\Phi - W_0 -$ Flat Generalized Sasakian-space-form:

Definition 3. A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is called $\Phi - W_0 -$ Flat if it satisfies

$$\Phi^2 A(\Phi X, \Phi Y)\Phi Z = 0$$

for any vector field X, Y, Z on the manifold[3].

Theorem 5. Every $W_0 -$ flat generalized Sasakian-space-form is $\Phi - W_0 -$ flat but the converse may not be true.

Proof. We prove that for a generalized Sasakian-space-form of dimension greater than three, the converse also holds.

Let us consider a $\Phi - W_0 -$ flat generalized Sasakian-space-form. Then by definition

$$\Phi^2 A(\Phi X, \Phi Y)\Phi Z = 0$$

In view of (2.14), the above equation yields

$$\Phi^2\{R(\Phi X, \Phi Y)\Phi Z - \frac{1}{2n}(S(\Phi Y, \Phi Z)\Phi X - g(\Phi X, \Phi Z)Q(\Phi Y))\} = 0$$

Using $(2.2), (2.6)$ & (2.7), we obtain from above

$$\begin{aligned}&\Phi^2\{f_1(g(\Phi Y, \Phi Z)\Phi X - g(\Phi X, \Phi Z)\Phi Y)\\ &+ f_2(g(\Phi X, \Phi^2 Z)\Phi^2 Y - g(\Phi Y, \Phi^2 Z)\Phi^2 X + 2g(\Phi X, \Phi^2 Y)\Phi^2 Z)\}\\ &= \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\Phi^2(g(\Phi Y, \Phi Z)\Phi X - g(\Phi X, \Phi Z)\Phi Y)\end{aligned} \tag{7.1}$$

Applying (2.3), we get from the above equation

$$\begin{aligned}&\Phi^2\{f_1(g(Y,Z)\Phi X - \eta(Y)\eta(Z)\Phi X - g(X,Z)\Phi Y + \eta(X)\eta(Z)\Phi Y)\\ &+ f_2(g(X, \Phi Z)\Phi^2 Y - g(Y, \Phi Z)\Phi^2 X + 2g(X, \Phi Y)\Phi^2 Z)\}\\ &= \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\Phi^2(g(Y,Z)\Phi X - \eta(Y)\eta(Z)\Phi X\\ &- g(X,Z)\Phi Y + \eta(X)\eta(Z)\Phi Y)\end{aligned}$$

By virtue of (2.1) and (2.2), the above equation yields

$$f_1(g(Y,Z)\Phi X - \eta(Y)\eta(Z)\Phi X - g(X,Z)\Phi Y + \eta(X)\eta(Z)\Phi Y)$$

$$+ f_2(g(X,\Phi Z)\Phi^2 Y - g(Y,\Phi Z)\Phi^2 X + 2g(X,\Phi Y)\Phi^2 Z)$$

$$= \frac{1}{2n}(2nf_1 + 3f_2 - f_3)(g(Y,Z)\Phi X - \eta(Y)\eta(Z)\Phi X$$

$$- g(X,Z)\Phi Y + \eta(X)\eta(Z)\Phi Y)$$

In the above equation, taking the inner product g in both sides with respect to W, we get

$$f_1(g(Y,Z)g(\Phi X,W) - \eta(Y)\eta(Z)g(\Phi X,W) - g(X,Z)g(\Phi Y,W) + \eta(X)\eta(Z)g(\Phi Y,W)$$

$$+ f_2(g(X,\Phi Z)g(\Phi^2 Y,W) - g(Y,\Phi Z)g(\Phi^2 X,W) + 2g(X,\Phi Y)g(\Phi^2 Z,W))$$

$$= \frac{1}{2n}(2nf_1 + 3f_2 - f_3)(g(Y,Z)g(\Phi X,W)$$

$$- \eta(Y)\eta(Z)g(\Phi X,W) - g(X,Z)g(\Phi Y,W) + \eta(X)\eta(Z)g(\Phi Y,W))$$

Putting $Y = Z = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, i = 1,2,......,2n+1,$ we get

$$3f_2 g(\Phi X,W) = \frac{3f_2 - f_3}{2n}(2n-1)g(\Phi X,W) \tag{7.2}$$

The above equation is true for any vector fields X and W. Let $W \neq X$. Then, it follows from the above equation that

$$3f_2 = \frac{3f_2 - f_3}{2n}(2n-1)$$

The above result implies

$$f_3 = \frac{3f_2}{1-2n}$$

From[12], it is known that a generalized Sasakian-space-form of dimension greater than three is W_0- flat if and only if $f_3 = \frac{3f_2}{1-2n}$. Hence, we see that a $\Phi - W_0 -$ flat generalized Sasakian-space-form is $W_0 -$ flat. Conversely, if the manifold is $W_0 -$ flat, then $A(X,Y)Z = 0$. From which it trivially follows that $\Phi^2 A(\Phi X, \Phi Y)\Phi Z = 0.$ Therefore, the manifold is $\Phi - W_0 -$ flat. □

8 Generalized Sasakian-space-form satisfying $A.R = 0$

Theorem 6. A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ *satisfies the condition*

$$A(\xi, X).R = 0$$

if and only if the functions f_2 & f_3 either satisfy the condition $(2n-1)f_3 + 3f_2 = 0$ or it has the sectional curvature $(f_1 - f_3)$.

Proof. The condition $A(\xi, X).R = 0$ yields to

$$A(\xi, X)R(Y,Z)U - R(A(\xi, X)Y, Z)U - R(Y, A(\xi, X)Z)U - R(Y,Z)A(\xi, X)U = 0 \tag{8.1}$$

for any vector fields X, Y, Z, U on M. In view of (2.19), we obtain

$$A(\xi, X)R(Y,Z)U = \frac{1}{2n}((1-2n)f_3 - 3f_2)\{g(X, R(Y,Z)U)\xi - (f_1 - f_3)(g(Z,U)\eta(Y) - g(Y,U)\eta(Z))X\} \tag{8.2}$$

On the other hand, by direct calculations, we have

$$R(A(\xi, X)Y, Z)U = \frac{1}{2n}((1-2n)f_3 - 3f_2)\{(f_1 - f_3)g(X,Y)[g(Z,U)\xi - \eta(U)Z] - \eta(Y)R(X,Z)U\} \tag{8.3}$$

and

$$R(Y, A(\xi, X)Z)U = \frac{1}{2n}((1-2n)f_3 - 3f_2)\{(f_1 - f_3)g(X,Z)[\eta(U)Y - g(Y,U)\xi] - \eta(Z)R(Y,X)U\} \tag{8.4}$$

and

$$R(Y,Z)A(\xi, X)U = \frac{1}{2n}((1-2n)f_3 - 3f_2)\{(f_1 - f_3)g(X,U)[\eta(Z)Y - \eta(Y)Z] - \eta(U)R(Y,Z)X\} \tag{8.5}$$

Substituting $(8.2), (8.3), (8.4)$ and (8.5) in (8.1), we arrive at

$$\begin{aligned}&\frac{1}{2n}((1-2n)f_3 - 3f_2)\{g(X, R(Y,Z)U)\xi - (f_1 - f_3)g(Z,U)\eta(Y)X\\&+(f_1 - f_3)g(Y,U)\eta(Z)X - (f_1 - f_3)g(X,Y)g(Z,U)\xi + (f_1 - f_3)g(X,Y)\eta(U)Z\\&+\eta(Y)R(X,Z)U - (f_1 - f_3)\eta(U)g(X,Z)Y + (f_1 - f_3)g(X,Z)g(Y,U)\xi\\&+\eta(Z)R(Y,X)U - (f_1 - f_3)g(X,U)\eta(Z)Y + (f_1 - f_3)g(X,U)\eta(Y)Z + \eta(U)R(Y,Z)X\}\\&= 0\end{aligned}$$

Taking inner product with ξ, above equation implies that

$$\frac{1}{2n}((1-2n)f_3 - 3f_2)\{g(X, R(Y,Z)U) - (f_1 - f_3)(g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\} = 0$$

There exist two cases. Either

$$g(X, R(Y,Z)U) - (f_1 - f_3)\{g(X,Y)g(Z,U) - g(X,Z)g(Y,U)\} = 0$$

which say us $M^{2n+1}(f_1, f_2, f_3)$ has the sectional curvature $(f_1 - f_3)$ or

$$(1-2n)f_3 - 3f_2 = 0$$

which implies that

$$(2n-1)f_3+3f_2=0$$ □

9 Generalized Sasakian-space-form satisfying $A.\tilde{C}=0$

Theorem 7. A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ *satisfies the condition*

$$A(\xi, X).\tilde{C}=0$$

if and only if either the $(2n-1)f_3+3f_2=0$ or $M^{2n+1}(f_1, f_2, f_3)$ is a real space form with the sectional curvature (f_1-f_3).

Proof. The condition $A(\xi, X).\tilde{C}=0$ implies that

$$(A(\xi, X)\tilde{C})(Y, Z, U)=A(\xi, X)\tilde{C}(Y, Z)U-\tilde{C}(A(\xi, X)Y, Z)U$$

$$-\tilde{C}(Y, A(\xi, X)Z)U-\tilde{C}(Y, Z)A(\xi, X)U \tag{9.1}$$

for any vector fields X, Y, Z on M. From (2.20) and (2.21), we can easily to see that

$$A(\xi, X)\tilde{C}'(Y, Z)U=\frac{1}{2n}[(1-2n)f_3-3f_2]\{g(\tilde{C}(Y, Z)U, X)\xi$$

$$-(f_1-f_3-\frac{r}{2n(2n+1)})(g(Z, U)\eta(Y)-g(Y, U)\eta(Z))X\} \tag{9.2}$$

$$\tilde{C}(A(\xi, X)Y, Z)U=\frac{1}{2n}[(1-2n)f_3-3f_2][(f_1-f_3-\frac{r}{2n(2n+1)})\{g(X, Y)$$

$$g(Z, U)\xi-g(X, Y)\eta(U)Z\}-\eta(Y)\tilde{C}(X, Z)U] \tag{9.3}$$

$$\tilde{C}(Y, A(\xi, X)Z)U=\frac{1}{2n}[(1-2n)f_3-3f_2][g(X, Z)\{(f_1-f_3-\frac{r}{2n(2n+1)})$$

$$(\eta(U)Y-g(Y, U)\xi)\}-\eta(Z)\tilde{C}(Y, X)U] \tag{9.4}$$

and

$$\tilde{C}(Y, Z)A(\xi, X)U=\frac{1}{2n}[(1-2n)f_3-3f_2][(f_1-f_3-\frac{r}{2n(2n+1)})(g(X, U)$$

$$\eta(Z)Y-g(X, U)\eta(Y)Z)-\eta(U)\tilde{C}(Y, Z)X] \tag{9.5}$$

Thus, substituting $(9.2), (9.3), (9.4)$ & (9.5) *in* (9.1), we get

$$\frac{1}{2n}[(1-2n)f_3-3f_2][g(\tilde{C}(Y, Z)U, X)\xi-(f_1-f_3-\frac{r}{2n(2n+1)})\{g(Z, U)\eta(Y)X$$

$$-g(Y, U)\eta(Z)X+g(X, Y)g(Z, U)\xi-g(X, Y)\eta(U)Z+g(X, Z)\eta(U)Y$$

$$-g(X,Z)g(Y,U)\xi+g(X,U)\eta(Z)Y-g(X,U)\eta(Y)Z\}+\eta(Y)\tilde{C}(X,Z)U$$
$$+\eta(Z)\tilde{C}(Y,X)U+\eta(U)\tilde{C}(Y,Z)X]=0$$

Taking inner product with ξ, above equation takes form

$$\frac{1}{2n}[(1-2n)f_3-3f_2][g(R(Y,Z)U,X)-(f_1-f_3)\{g(X,Y)g(Z,U)-g(X,Z)g(Y,U)\}]=0$$

This equation tells us that either $M^{2n+1}(f_1,f_2,f_3)$ is real space form with sectional curvature (f_1-f_3) or $(2n-1)f_3+3f_2=0$

Conversely, if $M^{2n+1}(f_1,f_2,f_3)$ is either real space form with sectional curvature (f_1-f_3) or it has $(2n-1)f_3+3f_2=0,$ then we can see that equation (9.1) satisfied. □

10 References

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