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Bounds of the Group of Automorphisms of Compact Riemann Surface With Reference To The Point Group of Sulphur Molecule

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Abstract

It is observed that every finite group is isometric to the group of Automorphisms of some Compact Riemann surface of genus $g(\geq 2)$. Also found that the group of symmetries of a non-linear molecule (point group) is an excellent example of finite groups. In this paper we have considered the point group of Sulphur molecule S_8, which is a group of order 16. Then we prove that the bounds of the order of symmetry group of Automorphisms of Compact Riemann Surface on which the point group of S_8 acts as a group of Automorphism is $4(g-1)$, and the corresponding minimum value of g is 5, associated Fuchsian Group of signature is $\Delta(2,2,2,2,2)$.

Key words : Compact Riemann Surface, Fuchsian Group, Genus, Group of Automorphisms, Point group.

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Introduction

Every finite group can be realized as a group of automorphisms of some Compact Riemann surfaces of genus $g(\geq 2)$ [2] Schwarz proved that if S is a compact Riemann surface of genus g, then Aut(S) is (i) infinite for g=0,1 and (ii) finite for $g \geq 2$. Hurwitz[9,10] showed that for a compact Riemann surface S of genus g there exists an upper bound on the order of Aut(S) when it is finite ($g \geq 2$) and the order of Aut(S) cannot exceed $84(g-1)$ and is attained g = 3, which is called Hurwitz group, and also infinitely many values of g, but is not attained for g = 2,4,5,6. A.M. Macbeath, the initiator of the study of Automorphism of Compact Riemann Surfaces, formulated two broad problems as follows-

(A) Find all large groups of Automorphisms including Hurwitz groups.

(B) Determine bounds for different classes of finite groups.[5,6]

In this paper we consider a molecule S_8 (sulphur) and then determine the group of symmetries of this molecule. We observed that the molecule S_8 possess the point group D_{4d} of order 16. [1] Considering this point group we determine a set of necessary and sufficient condition for existence of smooth epimorphism from a Fuchsian group Γ to the point group associated with S_8, which has appeared in a separate paper. [8] Following these conditions in this paper we shall determine the upper bound for the order of the point group D_{4d} acting on compact Riemann surface of genus g.

The point group D_{4d} of Sulphur molecule is generated by two elements a and b of orders 2 and 8 respectively, whose product has order 2.That is

$$D_{4d} = \langle a, b | \, a^2 = b^8 = (ab)^2 = 1 \rangle \tag{1}$$

The concept of Fuchsian group is directly related with the Automorphism groups of compact Riemann surfaces. The Fuchsian group Γ is an infinite group generated by k elements $x_1, x_2, x_3, \ldots \ldots, x_k$ of finite orders $m_1, m_2, m_3, \ldots \ldots, m_k$ respectively and 2γ elements $\alpha_1, \beta_1, \alpha_2\beta_2, \ldots \ldots, \alpha_\gamma\beta_\gamma$ of infinite order having presentation of the form:

$$\langle x_1, x_2, x_3, \ldots \ldots, x_k; \alpha_1, \beta_1, \alpha_2\beta_2, \ldots \ldots, \alpha_\gamma\beta_\gamma; {x_1}^{m_1} = {x_2}^{m_2} = \cdots \ldots \ldots = {x_k}^{m_k} = \prod_{i=1}^{k} x_i \prod_{j=1}^{\gamma} [\alpha_j\beta_j] = 1 \rangle. \tag{2}$$

Where $[\alpha_j\beta_j] = {\alpha_j}^{-1}{\beta_j}^{-1}\alpha_j\beta_j$, the commutator of $\alpha_j\beta_j$

And the measure $\delta(\Gamma) = 2\gamma - 2 + \sum(1 - \frac{1}{m_i}) > 0$ (3)

Generally Γ is denoted by $\Delta(\gamma; m_1, m_2, m_3, \ldots \ldots, m_k)$, called signature of Γ, the non-negative integers $m_1, m_2, m_3, \ldots \ldots, m_k$ are called periods and γ the genus of Γ. If k=0 then $\Gamma = (\gamma; -)$ is called a surface group i.e. a Fuchsian group which does not contain elements of finite order except identity. Thus a surface group is generated by 2g elements of infinite order $\alpha_1, \beta_1, \alpha_2\beta_2, \ldots \ldots, \alpha_g\beta_g$ satisfying the relations

$\prod_{i=1}^{g} [\alpha_i\beta_i] = 1$, where g is the genus of the group.

If $\Gamma = (\gamma; m_1, m_2, m_3, \ldots \ldots, m_k)$, then we use the expression

$$\Delta(\Gamma) = 2\pi\{2\gamma - 2 + \sum(1 - \frac{1}{m_i})\} \tag{4}$$

In particular, if K is a surface group of genus g, then $\Delta(K) = 4\pi(g-1)$ (5)

It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$$[\Gamma : \Gamma_1] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)} \tag{6}$$

If $\Gamma = (\gamma; m_1, m_2, m_3, \ldots \ldots, m_k)$ and K is a normal subgroup of Γ of genus g, then the surface kernel factor group Γ/K is an automorphism group to a compact Riemann surface of genus g [2] and

$$[\Gamma/K] = [\Gamma\text{:K}] = \frac{\Delta(K)}{\Delta(\Gamma)} = 4\pi(g-1) / \, 2\pi\{2\gamma - 2 + \sum(1 - \frac{1}{m_i})\} \tag{7}$$

A homomorphism from a Fuchsian group Γ to a finite group D_{4d} is said to be smooth if the kernel is a surface group of Γ. A factor group Γ/K where K is a surface genus g is called smooth quotient of genus g.

Theorem : There is a smooth epimorphism $\phi: \Gamma \to D_{4d}$ if and only if the following conditions are satisfied- [8]

1. When k=0 i.e. $\Gamma = \Delta(\gamma; -)$ surface group then $\gamma \geq 2$
2. When $k \neq 0$ then $0(\phi(x_i)) = m_i$; m_i divides 16

Moreover

(i) If all $\phi(x_i) \in \langle b \rangle$ then divides 8 and $\gamma \geq 1$

(ii) If all $\phi(x_i) \in D_{4d} - \langle b \rangle$ then m_i =2, k is even and $\gamma \geq 1$

(iii) If $\phi(x_i) \in \langle b \rangle$, $i = 1,2, \dots \dots s$; $s \geq k$ and

$\phi(x_{s+j}) \in D_{4d} - \langle b \rangle, j = 1,2, \dots \dots t$; then $s + t = k$

t is even and also $sl \equiv 0 (mod\ 8)$ and $1 \leq l \leq 7$. [8]

The main result:

Let S be a Compact Riemann surface of genus g admitting D_{4d} as its automorphism group. Then there exists a Fuchsian group Γ such that $D_{4d} \cong \Gamma / K$ where K is a surface group of genus g, i.e K is without any elements of finite order except the identity. So

$$\frac{2(g-1)}{0(D_{4d})} = 2\gamma - 2 + \sum_{i=1}^{k} 1 - \frac{1}{m_i} \tag{8}$$

Or, $\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i})$

By using the conditions of above mentioned theorem, it can be minimized the right side of the equation (8) so that we get the minimum value of g for which S admits D_{4d} as an automorphism group. Then we get-

Case (I) : $k = 0$

Then $\Gamma = \Delta(\gamma : -)$ is a surface group then $\gamma \geq 2$ then from (8)

$$\frac{g-1}{8} = 2\gamma - 2$$

$$\Rightarrow g \geq 17 \tag{9}$$

Case (II): $k \neq 0$

(i) If all $\phi(x_i) \in \langle b \rangle$ then m_i divides 8 and $\gamma \geq 1$, k is even i.e. m_i = 2,4,8

(a) k=2, m_i = 2 then

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{2}(1 - \frac{1}{m_i})$$

$$\Rightarrow g \geq 9 \tag{10}$$

(b) k=2, m_i = 4 then

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{2}(1 - \frac{1}{m_i})$$

$$\Rightarrow g \geq 13 \tag{11}$$

(c) k=2, m_i = 8 then

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{2}(1 - \frac{1}{m_i})$$

$\Rightarrow g \geq 15$ (12)

(ii) If all $\phi(x_i) \in D_{4d} - \langle b \rangle$ then m_i =2, k is even and $\gamma \geq 1$

(a) for k=2 we get

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{2}(1 - \frac{1}{m_i})$$

$\Rightarrow g \geq 9$ (13)

(b) for k=4 we get

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{4}(1 - \frac{1}{m_i})$$

$\Rightarrow g \geq 17$ (14)

(iii) If $\phi(x_i) \in \langle b \rangle$, and remaining $\phi(x_i) \in D_{4d} - \langle b \rangle$

i.e. $\phi(x_i) \in \langle b \rangle$ $i = 1,2, \ldots \ldots s < k$; and

$\phi(x_{s+j}) \in D_{4d} - \langle b \rangle, j = 1,2, \ldots \ldots t < k$; such that $s + t = k$

Here k≥3 as t is even and $s \neq 0$, $t \neq 0$.

(a) k=3 , s=1,t=2; $m_i = 2,4,8$; $m_j = 2$;$\gamma \geq 0$

(1) $m_i = 2$, $m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1 - \frac{1}{m_i}) + \sum_{j=1}^{t}(1 - \frac{1}{m_j})$$

$\Rightarrow g \geq -3$ (15)

(2) $m_i = 4$, $m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1 - \frac{1}{m_i}) + \sum_{j=1}^{t}(1 - \frac{1}{m_j})$$

$\Rightarrow g \geq -1$ (16)

(3) $m_i = 8$, $m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1 - \frac{1}{m_i}) + \sum_{j=1}^{t}(1 - \frac{1}{m_j})$$

$\Rightarrow g \geq 0$ (17)

(b) k=4 , s=2,t=2; $m_i = 2,4,8$; $m_j = 2$; $\gamma \geq 0$

(1) $m_i = 2$, $m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1 - \frac{1}{m_i}) + \sum_{j=1}^{t}(1 - \frac{1}{m_j})$$

$\Rightarrow g \geq 1$ (18)

(2) $m_i = 4$, $m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1 - \frac{1}{m_i}) + \sum_{j=1}^{t}(1 - \frac{1}{m_j})$$

$\Rightarrow g \geq 5$ (19)

(3) $m_i = 8, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 7 \tag{20}$$

(c) k=5 ,(s=1, t=4) or (s=3, t=2); $m_i = 2,4,8$, $m_j = 2$; $\gamma \geq 0$

(i) k=5 , s=1, t=4; $m_i = 2,4,8$; $m_j = 2$; $\gamma \geq 0$

(1) $m_i = 2, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 5 \tag{21}$$

(2) $m_i = 4, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 7 \tag{22}$$

(3) $m_i = 8, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 8 \tag{23}$$

(ii) k=5 , s=3,t=2; $m_i = 2,4,8$; $m_j = 2$; $\gamma \geq 0$

(1) $m_i = 2, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 5 \tag{24}$$

(2) $m_i = 4, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 11 \tag{25}$$

(3) $m_i = 8, m_j = 2$

$$\frac{2(g-1)}{16} = 2\gamma - 2 + \sum_{i=1}^{s}(1-\frac{1}{m_i}) + \sum_{j=1}^{t}(1-\frac{1}{m_j})$$

$$\Rightarrow g \geq 14 \tag{26}$$

From the above discussion we see that the minimum value of g will be obtained for k=5, where k is non-zero. Hervey stated that minimum value of g is ≥ 2 and then we get a finite automorphism group. So we have to consider the minimum value of g is 5 which is attained for k=5. It is clear that the Fuchsian groups corresponding to the lower bound is $\Delta(2,2,2,2,2)$.

Next we find an upper bound for the order of the point group D_{4d} acting on a Compact Riemann Surface of genus g

$$\frac{2(g-1)}{0(D_{4d})} \geq -2 + \left\{\left(1-\frac{1}{2}\right) + \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{2}\right)\right\} + \left\{\left(1-\frac{1}{2}\right) + \left(1-\frac{1}{2}\right)\right\}$$

$$\frac{2(g-1)}{0(D_{4d})} \geq 1/2$$

$$\Rightarrow 0(D_{4d}) \leq 4(g-1).$$

Conclusion

The maximum bound of the automorphism group of compact Riemann surface on which the symmetry group of the molecule S_8 acts as a group of automorphism is 4(g-1) which is attained for g=5 and k=5. The result can be summarized as follows:

Theorem : An upper bound for the order of the point group D_{4d} of the sulphur molecule S_8 acting on a compact Riemann surface of genus g is 4(g-1) where the minimum genus is 5, for k=5 and the corresponding Fuchsian group has the signature Δ(5:2,2,2,2.2).

It is observed that this result follows the result established by Bujalance in his paper "On Compact Riemann Surfaces with Dihedral Groups of Automorphism."[4]

Future Prospect: The study of Compact Riemann Surfaces with reference to various classes of finite groups had been carried out by different researchers during last decade of 20th centuries. It is very interesting to study the topic with reference to a physical molecule. There are several molecules whose group of symmetries are not included in the classes of finite groups which are already studied. Therefore the potential for future study the topic with reference to different molecules is very high.

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