



(Print)

Section A

(Online)



Estd. 1989

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
 An International Open Free Access Peer Reviewed Research Journal of Mathematics
 website:- www.ultrascientist.org

Generalized Convolution of Certain Class of Harmonic Univalent Functions

BALVIR SINGH¹ and SAURABH PORWAL²

¹Assoc. Professor, Department of Mathematics
 R P Degree College, Kamalganj, Farrukhabad-209601 (U.P.) (India)
 e-mail: balvirsingh.rp@gmail.com

²Lecturer Mathematics Sri Radhey Lal Arya Inter College, Aihan, Hathras (U.P.) (India)

Corresponding Author e-mail: saurabhjcb@rediffmail.com

<http://dx.doi.org/10.22147/jusps-A/300101>

Acceptance Date 15th Nov., 2017,

Online Publication Date 2nd January, 2018

Abstract

The purpose of the present paper is to establish some results generalized convolution for a subclass of harmonic univalent functions.

Key words : Harmonic, Univalent, Convolution.

AMS 2010 Mathematics Subject Classifications: 30C45.

1 Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected complex domain D if both u and v are real harmonic in D . In any simply-connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$, (see Clunie and Sheil-Small³). For more basic results on harmonic functions one may refer following standard introductory text books by Duren⁹, (see also Ahuja¹ and Ponnusamy and Rasila¹²).

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$, we

may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (1.1)$$

A function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \bar{z}^n \quad (1.2)$$

in S_H is said to be in the class $R_H(\beta)$ if and only if

$$\operatorname{Re}\{h'(z) + g'(z)\} < \beta, \quad z \in U \quad (1.3)$$

for some $\beta(1 < \beta \leq 2)$.

The class $R_H(\beta)$ has been extensively studied by Dixit and Porwal⁶.

Let $f_j(z)$ ($j=1,2$) in S_H be given by

$$f_j(z) = z + \sum_{n=2}^{\infty} |a_{n,j}| z^n + \sum_{n=1}^{\infty} |b_{n,j}| \bar{z}^n. \quad (1.4)$$

Then the convolution $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} |a_{n,1} a_{n,2}| z^n + \sum_{n=1}^{\infty} |b_{n,1} b_{n,2}| \bar{z}^n. \quad (1.5)$$

Furthermore, for any real number p and q , we define the generalized convolution $(f_1 \Delta f_2)(p, q; z)$ by

$$(f_1 \Delta f_2)(p, q; z) = z + \sum_{n=2}^{\infty} |a_{n,1}|^p |a_{n,2}|^q z^n + \sum_{n=1}^{\infty} |b_{n,1}|^p |b_{n,2}|^q \bar{z}^n. \quad (1.6)$$

In the special case, if we take $p=q=1$, then we have

$$(f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z), \quad (z \in U). \quad (1.7)$$

Study of convolution play an important role in Geometric Function Theory. It has attracted large number of researchers of the field. By making use of convolution, several new and interesting subclasses have been defined and studied in the direction of Subordination, Partial sums, Neighbourhood, Argument Problems, Integral mean inequalities and some other related interesting properties. For detailed study see the excellent text book by Ruscheweyh¹⁶. In 1975, Schild and Silverman¹⁷ studied the various interesting results on the convolution of analytic functions. Later on, Choi *et al.*², Darwish⁴, Darwish and Aouf⁵, Nishiwaki and Owa¹⁰, Owa¹¹ studied the generalized convolution for analytic functions only. Very recently, Porwal and Dixit^{13, 14} (see also^{7, 8, 15, 18}) studied the analogous results on harmonic univalent functions. In the present paper, we study systematically on the generalized convolution of harmonic univalent functions for the class $R_H(\beta)$.

2 Main Results

In order to prove our results for functions to the class $R_H(\beta)$, we shall need the following lemma given by Dixit and Porwal⁶.

Lemma 2.1. Let the function $f(z)$ be defined by (1.2). Then $f \in R_H(\beta)$, if and only if

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq \beta - 1, \quad (2.1)$$

where $1 < \beta \leq 2$.

Theorem 2.1. If the function $f_j(z)$ ($j = 1, 2$) defined by (1.4) with $b_{1,j} = 0$ ($j = 1, 2$) are in the classes $R_H(\beta_j)$ for each j and the condition

$$|a_{n,1}|^{\frac{1}{p}}|a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}}|b_{n,2}|^{\frac{1}{q}} \leq (|a_{n,1}| + |b_{n,1}|)^{\frac{1}{p}}(|a_{n,2}| + |b_{n,2}|)^{\frac{1}{q}}, \quad (2.2)$$

for $(n = 2, 3, \dots)$ is satisfied then

$$(f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in R_H(\beta), \quad (2.3)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta = 1 + (\beta_1 - 1)^{\frac{1}{p}}(\beta_2 - 1)^{\frac{1}{q}}$.

Proof: Since $f_j(z) \in R_H(\beta)$, by using Lemma 2.1, we have

$$\sum_{n=2}^{\infty} \left(\frac{n}{\beta_j - 1} \right) (|a_{n,j}| + |b_{n,j}|) \leq 1, \quad (j = 1, 2). \quad (2.4)$$

From (2.4) we have

$$\left\{ \sum_{n=2}^{\infty} \left(\frac{n}{\beta_1 - 1} \right) (|a_{n,1}| + |b_{n,1}|) \right\}^{\frac{1}{p}} \leq 1, \quad (2.5)$$

and

$$\left\{ \sum_{n=2}^{\infty} \left(\frac{n}{\beta_2 - 1} \right) (|a_{n,2}| + |b_{n,2}|) \right\}^{\frac{1}{q}} \leq 1. \quad (2.6)$$

Now

$$\begin{aligned}
& \sum_{n=2}^{\infty} \left(\frac{n}{\beta_1 - 1} \right)^{\frac{1}{p}} \left(\frac{n}{\beta_2 - 1} \right)^{\frac{1}{q}} \left(|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \right) \\
& \leq \sum_{n=2}^{\infty} \left(\frac{n}{\beta_1 - 1} \right)^{\frac{1}{p}} \left(\frac{n}{\beta_2 - 1} \right)^{\frac{1}{q}} \left(|a_{n,1}| + |b_{n,1}| \right)^{\frac{1}{p}} \left(|a_{n,2}| + |b_{n,2}| \right)^{\frac{1}{q}} \quad (\text{Using (2.2)}) \\
& \leq \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{\beta_1 - 1} \right) \left(|a_{n,1}| + |b_{n,1}| \right) \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{\beta_2 - 1} \right) \left(|a_{n,2}| + |b_{n,2}| \right) \right\}^{\frac{1}{q}} \\
& \quad (\text{Using Hölder's Inequality}) \\
& \leq 1. \quad (\text{Using (2.5) and (2.6)})
\end{aligned}$$

Since

$$(f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) = z + \sum_{n=2}^{\infty} |a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} z^n + \sum_{n=2}^{\infty} |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \bar{z}^n. \quad (2.7)$$

It suffices to show that $(f_1 \Delta f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in R_H(\beta)$, if

$$\sum_{n=2}^{\infty} \left(\frac{n}{\beta - 1} \right) \left(|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \right) \leq 1. \quad (2.8)$$

For this we have to show that L.H.S. of (2.8) is bounded above by

$$\sum_{n=2}^{\infty} \left(\frac{n}{\beta_1 - 1} \right)^{\frac{1}{p}} \left(\frac{n}{\beta_2 - 1} \right)^{\frac{1}{q}} \left(|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \right),$$

which is equivalent to $\beta \geq 1 + (\beta_1 - 1)^{\frac{1}{p}} (\beta_2 - 1)^{\frac{1}{q}}$.

Theorem 2.2. Let the function $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (1.4) with $b_{1,j} = 0$ ($j = 1, 2, \dots, m$),

be in the classes $R_H(\beta_j)$ for each j and let $F_m(z)$ be defined by

$$F_m(z) = z + \sum_{n=2}^{\infty} \left(\sum_{j=1}^m |a_{n,j}|^p \right) z^n + \sum_{n=2}^{\infty} \left(\sum_{j=1}^m |b_{n,j}|^p \right) \bar{z}^n, \quad (p \geq 1). \quad (2.9)$$

Then $F_m(z) \in R_H(\beta_m)$, where $\beta_m = 1 + \frac{m(\beta-1)^p}{2^{p-1}}$, $\beta = \max_{1 \leq j \leq m} \beta_j$ and $m(\beta-1)^p \leq 2^{p-1}$.

Proof: Since $f_j(z) \in R_H(\beta_j)$, by using Lemma 2.1, we observe that

$$\sum_{n=2}^{\infty} \left(\frac{n}{\beta_j - 1} \right) \left(|a_{n,j}| + |b_{n,j}| \right) \leq 1, \quad (j = 1, 2, \dots, m) \quad (2.10)$$

and

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{n}{\beta_j - 1} \right)^p \left(|a_{n,j}|^p + |b_{n,j}|^p \right) \\ & \leq \sum_{n=2}^{\infty} \left(\frac{n}{\beta_j - 1} \right)^p \left(|a_{n,j}| + |b_{n,j}| \right)^p \\ & \leq \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{\beta_j - 1} \right) \left(|a_{n,j}| + |b_{n,j}| \right) \right\}^p \\ & \leq 1. \quad (\text{Using (2.10)}) \end{aligned} \quad (2.11)$$

It follows that from (2.11) that

$$\sum_{n=2}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^m \left(\frac{n}{\beta_j - 1} \right)^p \left(|a_{n,j}|^p + |b_{n,j}|^p \right) \right\} \leq 1. \quad (2.12)$$

Putting $\beta = \max_{1 \leq j \leq m} \beta_j$ and by virtue of Lemma 2.1, we find that

$$\sum_{n=2}^{\infty} \left(\frac{n}{\beta_m - 1} \right) \left(\sum_{j=1}^m |a_{n,j}|^p + |b_{n,j}|^p \right)$$

$$\sum_{n=2}^{\infty} \frac{1}{m} \left(\frac{n}{\beta-1} \right)^p \left(\sum_{j=1}^m \left(|a_{n,j}|^p + |b_{n,j}|^p \right) \right) \leq 1.$$

$$\text{if } \beta_m \geq 1 + \frac{m(\beta-1)^p}{n^{p-1}}, \quad (n \geq 2).$$

$$\text{Now let } g(n) = 1 + \frac{m(\beta-1)^p}{n^{p-1}}.$$

It is easy to verify that $g(n)$ is an decreasing function of n for $p \geq 1$.

Theorefore,

$$\beta_m = \sup_{n \geq 2} g(n) = g(2) = 1 + \frac{m(\beta-1)^p}{2^{p-1}}.$$

By $m(\beta-1)^p \leq 2^{p-1}$, we see that $1 < \beta \leq 2$.

This completes the proof of Theorem 2.2.

Acknowledgement

The authors are thankful to the referee for his valuable comments and observations which helped in improving the paper.

References

1. O.P. Ahuja, Planar harmonic univalent and related mappings, *J. Inequal. Pure Appl. Math.*, 6(4), Art. 122, 1-18 (2005).
2. J.H. Choi, Y.C. Kim and S. Owa, Generalizations of Hadamard products of functions with negative coefficients, *J. Math. Anal. Appl.*, 199, 495-501 (1996).
3. J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fen. Series AI Math.*, 9, 3-25 (1984).
4. H.E. Darwish, On generalizations of Hadamard products of functions with negative coefficients, *Proc. Pak. Acad. Sci.*, 43(4), 269-273 (2006).
5. H.E. Darwish and M.K. Aouf, Generalizations of modified-Hadamard products of p -valent functions with negative coefficients, *Math. Comput. Modell.*, 49, 38-45 (2009).
6. K.K. Dixit and S. Porwal, A subclass of harmonic univalent functions with positive coefficients, *Tamkang J. Math.*, 41(3), 261-269 (2010).
7. K.K. Dixit and S. Porwal, A convolution approach on partial sums of certain analytic and univalent functions, *J. Inequal. Pure Appl. Math.*, 10(4), Art. 101, 1-17 (2009).
8. K.K. Dixit and S. Porwal, Some properties of harmonic functions defined by convolution, *Kyungpook Math. J.*, 49(4), 751-761 (2009).
9. P. Duren, *Harmonic mappings in the plane*, Camb. Univ. Press, (2004).

10. J. Nishiwaki and S. Owa, An application of Hölder's inequality for convolutions, *J. Inequal. Pure Appl. Math.*, 10(4), Art. 98, 1-14 (2009).
11. S. Owa, The Quasi-Hadamard products of certain analytic functions, in: H.M. Srivastava, S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific publishing Company, Singapore, New Jersey, London, Hong Kong, 234-251 (1992).
12. S. Ponnusamy and A. Rasila, Planar harmonic mappings, *Ramanujan Mathematical society Mathematics Newsletters*, 17(2), 40-57 (2007).
13. S. Porwal and K.K. Dixit, Some properties of generalized convolution of harmonic univalent functions, *Demonstratio Mathematica.*, 46(1), 63-74 (2013).
14. Saurabh Porwal and K.K. Dixit, Convolution on a Generalized class of harmonic univalent functions, *Kyungpook Math. J.*, 55, 83-89 (2015).
15. Saurabh Porwal and M.V. Singh, Convolution on a certain class of harmonic univalent functions, *J. Ind. Math. Soc.*, 82(1-2), 117-127 (2015).
16. S. Ruscheweyh, *Convolutions in Geometric Function theory*, Sem. Math. Sup., 83 Presses Univ. de Montreal, (1982).
17. A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska Sect. A* 29, 99-107 (1975).
18. R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, 106(1), 145-152 (1989).