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Bayesian Prediction limits for Rayleigh Model when observations are Multiply type II Censored

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Abstract

In this article, multiply type II censored observation from Rayleigh model is considered. Using conjugate prior for parameter, predictive pdf, hence and predictive limits are obtained. We also obtained prediction limits for double, mid censoring as well. A Monte Carlo study of 1000 randomly generated sample is performed and prediction limits for different censoring scheme is compared.

Key words : Rayleigh Model, prior, conjugate prior, posterior, Bayes prediction, Multiply type II censoring.

Mathematics Subject Classification : 62F15, 62N01, 00A72

1. Introduction

Rayleigh model which is special case of Weibull model has a wide application, such as in life testing experiments which rapidly age with time, as its failure rate is a liner function of time. In communication engineering Rayleigh model play important role and has been successfully used for radio active power distribution. The time to failure x , of a Rayleigh component has probability function

$$f(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad x > 0, \sigma > 0 \quad (1)$$

with cumulative distribution function

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad (2)$$

and the reliability function at time t

$$R(t) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \quad (3)$$

For larger t , reliability of a Rayleigh components decreases with time more rapidly than in the case of exponential distribution.

For making inference about future sample characteristics, predictive density is determined by combining the posterior distribution with the pdf of future characteristics given parameter. Integration with respect to each of parameter of this combination yields the predictive distribution for the future characteristics which summarizes the knowledge about future sample in the light of informations provided by the given data (see, Aitchison and Dunsmore)².

Bhattacharya and Tyagi (1990) mentioned that in some clinical studies dealing with cancer patients, the survival pattern follows the Rayleigh distribution. Dyer and Smith^{7,8} obtained Best linear unbiased estimator of the parameter of Rayleigh distribution. Sinha and Howlader (1983) obtained credible and HPD intervals for the parameter and reliability function of this distribution. Bayesian approach inference in reliability studies based on doubly type II data was presented by Fernandez⁹. Dey and Dey⁵ obtained inference for the Rayleigh model under progressively Type-II censoring with binomial removal. Abdel-Hamid and AL-Hussaini¹ obtained Bayesian prediction for type-II progressive-censored data from the Rayleigh distribution under progressive-stress model. Based on general progressive type II censoring Mousa and Sagheer (2006), Kim and Han¹⁰, Dey *et al.*⁶ dealt the model in Bayesian perspective.

Kim and Han¹⁰ obtained classical and Bayesian estimators when data is compounded with multiply type II censoring. Shastri *et al.* (2010) obtained Bayes prediction limits for exponential distribution for multiply type II censored data. Dey and Das⁴ discussed prediction interval for a Rayleigh distribution. But there appears to be nothing in the literature for Bayes prediction limits for Rayleigh model when observations are multiply type II censoring.

2. Censoring Scheme :

Multiply type II censored samples may arise in practice in a number of ways. They may arise, for example, in life testing experiments when the life times of some units are not observed due to mechanical or experimental difficulties. Another situation where multiply censored samples arise naturally is when some units failed between two points of observation with exact times to failures of these units unobserved. The multiply type II censoring was discussed by Balasubramanian and Balakrishnan³, Upadhyay *et al.*¹³ among others.

Out of N items put on test, the multiply type II censoring scheme supposes that first r , last s and middle l observations are censored and the only observations available are $x_{r+1} < \dots < x_{r+k}$ and $x_{r+k+l+1} < \dots < x_{N-s}$. If we substitute $r=l=0$ this censoring scheme reduces to type II right censoring scheme, on substituting $l=s=0$, censoring scheme reduces to type II right censoring scheme. This also reduces to a doubly type II censoring scheme when $l=0$, a reverse scheme named type II mid censoring appears at $r=s=0$.

3. Prediction limits :

Let us assume that x_1, x_2, \dots, x_N be a random sample of size N drawn from a Rayleigh model (1). Consider the multiply type II censoring scheme described in previous section and let $x_{r+1} < \dots < x_{r+k}$ and $x_{r+k+l+1} < \dots < x_{N-s}$ be the observed life times.

The likelihood function (LF) for this situation can be written as

$$L = \frac{N!}{r!l!s!} [F(x_{r+1})]^r [F(x_{r+k+l+1}) - F(x_{r+k})]^l [1 - F(x_{N-s})]^s \prod_{i=r+1}^{r+k} f(x_i/\sigma) \prod_{i=r+k+l+1}^{N-s} f(x_i/\sigma) \quad (4)$$

Using (1) and (3), and on simplification, it reduces to

$$L = \frac{N!}{r!l!s!} \left(\frac{1}{\sigma^2}\right)^A \sum_{p=0}^r \sum_{g=0}^l \Omega_p \Omega_g \cdot \exp\left[-\frac{S_c + S_b}{2\sigma^2}\right] \prod_{i=r+1}^{r+k} x_i \prod_{i=r+k+l+1}^{N-s} x_i \quad (5)$$

where

$$S_c = px_{r+1}^2 + (l - g)x_{r+k}^2 + gx_{r+k+l+1}^2$$

$$S_b = sx_{N-s}^2 + \sum_{i=r+1}^{r+k} x_i^2 + \sum_{i=r+k+l+1}^{N-s} x_i^2$$

$$\Omega_p = (-1)^p \binom{r}{p},$$

$$\Omega_g = (-1)^g \binom{l}{g}$$

and

$$A = N - r - l - s$$

Consider a conjugate family of prior for the parameter σ

$$g(\sigma/a, b) = \frac{a^b}{\Gamma(b)2^{b-1}} \sigma^{-2b-1} \exp\left[-\frac{a}{2\sigma^2}\right]; \quad \sigma > 0; a, b > 0 \quad (6)$$

Combining LF(5) with prior (6) via Bayes theorem, the posterior distribution is defined and obtained as

$$p(\sigma|\underline{x}) = \frac{L(\underline{x}, \sigma)g(\sigma)}{\int L(\underline{x}, \sigma)g(\sigma)d\sigma}$$

$$p(\sigma|\underline{x}) = \frac{1}{\Gamma(b+A)} \frac{1}{2^{b+A-1}} \frac{\sum_{p=0}^r \sum_{g=0}^l \Omega_p \Omega_g \sigma^{-2(A+b)-1} \exp\left(-\frac{S_c + S_b + a}{2\sigma^2}\right)}{\sum_{p=0}^r \sum_{g=0}^l \Omega_p \Omega_g (S_c + S_b + a)} \quad (7)$$

Let y_1, y_2, \dots, y_m be the second independent random sample of size m of future observation from the model (1), then the density of n^{th} future observation will be obtained by

$$f(y|\sigma) = \frac{m!}{(n-1)!(m-n)!} [F(y)]^{n-1} f(y) [1 - F(y)]^{m-n}$$

Substituting and solving

$$f(y|\sigma) = \beta^{-1}(n, m - n + 1) \sum_{i=0}^{n-1} \Omega_i \exp\left[-\frac{1}{2\sigma^2} (M + i)y^2\right] \left(\frac{y}{\sigma^2}\right) \quad (8)$$

where

$$\beta^{-1}(n, m - n + 1) = \frac{m!}{(n-1)!(m-n)!}$$

and

$$M = m - n + 1$$

Then the Bayes Predictive density for future n^{th} order observation will be

$$h(y|\underline{x}) = \int f(y|\sigma)p(\sigma|\underline{x}) d\sigma$$

substituting the values

$$h(y|\underline{x}) = \frac{2(b+A)\beta^{-1}(n, m-n+1)}{C(x)} \sum_{p=0}^r \sum_{g=0}^l \sum_{i=0}^{n-1} \Omega_p \Omega_g \Omega_i [S_c + S_b + a + (M+i)y]^{-(A+b+1)} y$$

where

$$C(x) = \sum_{p=0}^r \sum_{g=0}^l \Omega_p \Omega_g [S_c + S_b + a]^{-(A+b)}$$

In the context of Bayes prediction, we say here that (t_{1n}, t_{2n}) is a $100(1-\alpha)\%$ limit for future n^{th} ordered random variable, if

$$P_r[t_{1n} \leq y_{(n)} \leq t_{2n}] = 1 - \alpha$$

Here t_{1n} and t_{2n} are said to be lower and upper Bayes prediction limit for n^{th} ordered random variable $y_{(n)}$ and $(1-\alpha)$ is called the confidence prediction coefficient. One-sided Bayes Prediction bound limits are obtained by solving

$$P_r[y_{(n)} \leq t_{1n}] = \frac{\alpha}{2} = P_r[y_{(n)} \geq t_{1n}]$$

Above can be rewritten as

$$P_r[y_{(n)} \leq t_{1n}] = \int_0^{t_{1n}} h(y|\underline{x}) dy = \frac{\alpha}{2} \quad (9)$$

and

$$P_r[y_{(n)} \leq t_{2n}] = \int_0^{t_{2n}} h(y|\underline{x}) dy = 1 - \frac{\alpha}{2} \quad (10)$$

Using (8), (9) and (10), the one sided Bayes Prediction bound limits are obtained by solving

$$\frac{\beta^{-1}(n, M)}{C(x)} \sum_{p=0}^r \sum_{g=0}^l \sum_{i=0}^{n-1} \Omega_p \Omega_g \Omega_i \frac{1}{(M-i)} [(S_c + S_b + a)^{-(A+b)} - (S_c + S_b + a + (M+i)t_{1n}^2)^{-(A+b)}] = \frac{\alpha}{2} \quad (11)$$

and

$$\frac{\beta^{-1}(n, M)}{C(x)} \sum_{p=0}^r \sum_{g=0}^l \sum_{i=0}^{n-1} \Omega_p \Omega_g \Omega_i \frac{1}{(M-i)} [(S_c + S_b + a + (M+i)t_{2n}^2)^{-(A+b)}] = \frac{\alpha}{2} \quad (12)$$

4. Discussion

For numerical illustration we have computed prediction interval for the smallest order future observation on the basis of 1000 samples each of size 10 and 20 generated by Monte Carlo simulation technique.

A random sample of size $N=10$ was generated from the distribution in (1) for different values of σ . In order to obtain the prediction intervals for first observation of a future sample of size $m=10$ at a nominal 95% prediction level choosing $m=10, 20$ for $N=10$ and 20 respectively. Different combination of r, k, l and s are taken considering different censoring fraction. The effects of hyperparameters $a = (0.01, 0.1, 1.0, 2.0, 4.0)$ and $b = (1.0, 2.0, 4.0)$ was also considered separately.

When $N=m=20$, it can be observed from the tables 4-6, at 95% confidence level, as censored sample size decreases prediction intervals become shorter. Prediction limits are found shortest at $r=6, k=3, l=6, s=4$. On comparing different censoring scheme, namely Multiply, right, left, doubly and mid, it is evident from tables that multiply type II censoring scheme outperforms other censoring schemes.

Effect of hyperparameter a can be studied from each table. For fixed hyperparameter b , prediction intervals increases with an increase in a everywhere. A reverse case is noticed with the variation in hyperparameter b . With the increase in value of b , prediction intervals decreases.

Similar patterned result is found when size of informative as well as future sample is taken at $N=m=10$. The results are summarized in tables 1-3, at 95% confidence level. For different values of r, k, l and s , prediction intervals are reported with variation in a keeping b fixed. With more number of censored observations, prediction intervals found to be shorter. It can further be shorter if a is at minimum value. Keeping other values fixed, we observe that an increase in b provides shorter prediction interval.

Due to paucity of space tables for 99% prediction confidence is not reported over here, it was found that prediction intervals converges towards first order future observation only. Prediction intervals obtained with different constants have same affect but with shorter width. A further study on MCMC methods have scope to such problems, where data is compounded by complex censoring schemes.

Conclusion

Proposed form of prediction interval is recommended with larger sample size with higher prediction confidence. With small sample size multiply type II censoring can be taken under consideration with more number of censored observations is suggested. As far as prior hyperparameters are concern use of smaller value of a and larger value of b is suggested. Comparison with different censoring scheme suggests, the use of Multiply type II censoring scheme for obtaining lowest prediction interval.

Censoring Scheme	Table 1 Bayes Prediction limits at $b=1.0$ for sample size $N=m=20$ at $\sigma=2.0$					
	$r \ k \ l \ s$	$a=0.01$	$a=0.1$	$a=1.0$	$a=2.0$	$a=4.0$
Multiply	2 3 1 1	0.005776	0.005778	0.005791	0.005806	0.005835
	3 3 1 2	0.005719	0.00572	0.005734	0.00575	0.005782
	3 3 2 3	0.005428	0.00543	0.005445	0.005463	0.005498
	3 3 3 4	0.005247	0.005249	0.005267	0.005286	0.005325
	5 3 4 3	0.005378	0.00538	0.005397	0.005415	0.005451
	5 3 5 4	0.0051	0.005101	0.00512	0.005141	0.005182
	6 3 6 4	0.004967	0.004969	0.004989	0.00501	0.005053
Right	0 3 0 6	0.005443	0.005445	0.005464	0.005486	0.005528
Left	6 3 0 0	0.005656	0.005657	0.005671	0.005685	0.005714
Doubly	3 3 0 3	0.005414	0.005416	0.005432	0.00545	0.005485

Mid	0 3 6 0	0.005443	0.005445	0.005464	0.005486	0.005528
Table 2. Bayes Prediction limits at b=2.0 for sample size N=m=20 at sigma=2.0						
Censoring Scheme	$r \ k \ t \ s$	a=0.01	a=0.1	a=1.0	a=2.0	a=4.0
Multiply	2 3 1 1	0.005664	0.005666	0.005679	0.005693	0.005722
	3 3 1 2	0.005602	0.005604	0.005618	0.005633	0.005664
	3 3 2 3	0.005314	0.005316	0.005331	0.005348	0.005383
	3 3 3 4	0.005133	0.005135	0.005152	0.005171	0.005208
	5 3 4 3	0.005264	0.005266	0.005282	0.005299	0.005335
	5 3 5 4	0.004986	0.004988	0.005006	0.005026	0.005066
	6 3 6 4	0.004858	0.004859	0.004878	0.004899	0.004941
Right	0 3 0 6	0.005307	0.005309	0.005328	0.005348	0.005389
Left	6 3 0 0	0.005552	0.005553	0.005566	0.00558	0.005609
Doubly	3 3 0 3	0.005301	0.005303	0.005318	0.005335	0.00537
Mid	0 3 6 0	0.005594	0.005595	0.005608	0.005622	0.00565

Table 3 : Bayes Prediction limits at b=4.0 for sample size N=m=20 at sigma=2.0						
Censoring Scheme	$r \ k \ t \ s$	a=0.01	a=0.1	a=1.0	a=2.0	a=4.0
Multiply	2 3 1 1	0.005463	0.005465	0.005477	0.005491	0.005518
	3 3 1 2	0.005395	0.005397	0.00541	0.005424	0.005453
	3 3 2 3	0.005113	0.005114	0.005129	0.005145	0.005177
	3 3 3 4	0.004932	0.004933	0.004949	0.004967	0.005002
	5 3 4 3	0.005061	0.005063	0.005078	0.005095	0.005128
	5 3 5 4	0.004787	0.004788	0.004806	0.004825	0.004863
	6 3 6 4	0.004665	0.004667	0.004685	0.004705	0.004744
Right	0 3 0 6	0.005072	0.005074	0.005091	0.00511	0.005148
Left	6 3 0 0	0.005365	0.005366	0.005378	0.005392	0.005419
Doubly	3 3 0 3	0.0051	0.005102	0.005117	0.005133	0.005165
Mid	0 3 6 0	0.005404	0.005406	0.005418	0.005431	0.005458

Table 4. Bayes Prediction limits at b=1.0 for sample size N=m=10 at sigma=2.0						
Censoring Scheme	$r \ k \ t \ s$	a=0.01	a=0.1	a=1.0	a=2.0	a=4.0
Multiply	1 2 1 1	0.008196	0.008204	0.008278	0.008359	0.00852
	2 2 1 1	0.0082	0.008208	0.008282	0.008363	0.008524
	3 2 1 1	0.008199	0.008206	0.00828	0.008362	0.008523
	3 2 2 1	0.008207	0.008215	0.008289	0.008371	0.008532
	3 2 2 2	0.006797	0.006805	0.006886	0.006981	0.007175
	3 2 3 2	0.008707	0.008716	0.008805	0.008903	0.009094
Multiply	3 2 3 3	0.009407	0.009416	0.009507	0.009606	0.009799
Right	0 2 0 4	0.006928	0.006938	0.007043	0.007165	0.007413
Left	4 2 0 0	0.008409	0.008415	0.008481	0.008554	0.008697
Doubly	2 2 0 2	0.006796	0.006804	0.006885	0.006978	0.007171

Mid	0 2 4 0	0.00835	0.008357	0.008423	0.008496	0.008641
Censoring Scheme	Table 5. Bayes Prediction limits at b=2.0 for sample size N=m=10 at sigma=2.0					
	$r \ k \ t \ s$	a=0.01	a=0.1	a=1.0	a=2.0	a=4.0
	1 2 1 1	0.007942	0.007949	0.008018	0.008094	0.008245
	2 2 1 1	0.007946	0.007952	0.008022	0.008098	0.008248
	3 2 1 1	0.007944	0.007951	0.00802	0.008096	0.008247
	3 2 2 1	0.007951	0.007958	0.008028	0.008104	0.008255
	3 2 2 2	0.00666	0.006666	0.006733	0.006814	0.006984
	3 2 3 2	0.008345	0.008354	0.008438	0.008531	0.008711
Multiply	3 2 3 3	0.008956	0.008965	0.00905	0.009144	0.009327
Right	0 2 0 4	0.00674	0.006748	0.006833	0.006933	0.007146
Left	4 2 0 0	0.008163	0.008169	0.008231	0.008299	0.008435
Doubly	2 2 0 2	0.006659	0.006666	0.006733	0.006813	0.006981
Mid	0 2 4 0	0.008107	0.008113	0.008176	0.008245	0.008381

Censoring Scheme	Table 6. Bayes Prediction limits at b=4.0 for sample size N=m=10 at sigma=2.0					
	$r \ k \ t \ s$	a=0.01	a=0.1	a=1.0	a=2.0	a=4.0
	1 2 1 1	0.007538	0.007544	0.007605	0.007672	0.007805
	2 2 1 1	0.007541	0.007547	0.007608	0.007675	0.007808
	3 2 1 1	0.007539	0.007545	0.007606	0.007673	0.007806
Multiply	3 2 2 1	0.007545	0.007551	0.007612	0.007679	0.007813
	3 2 2 2	0.006484	0.006487	0.006527	0.006581	0.006705
	3 2 3 2	0.007783	0.00779	0.007866	0.007949	0.00811
	3 2 3 3	0.008264	0.008272	0.008349	0.008433	0.008597
Right	0 2 0 4	0.006519	0.006523	0.006568	0.006631	0.00678
Left	4 2 0 0	0.007763	0.007768	0.007824	0.007885	0.008006
Doubly	2 2 0 2	0.006483	0.006487	0.006527	0.006581	0.006705
Mid	0 2 4 0	0.007712	0.007717	0.007773	0.007835	0.007957

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