



(Print)

Section A

(Online)



Estd. 1989

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES
 An International Open Free Access Peer Reviewed Research Journal of Mathematics
 website:- www.ultrascientist.org

Note on Super Radial Signed Graphs

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<http://dx.doi.org/10.22147/jusps-A/300105>

Acceptance Date 01st December, 2017,

Online Publication Date 2nd January, 2018

Abstract

In this paper we introduced a new notion super radial signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterization of radial signed graphs. Further, we presented some switching equivalent characterizations.

Key words and Phrases : Signed graphs, Balance, Switching, Super radial signed graph, Negation of a signed graph.

2000 Mathematics Subject Classification: 05C22, 05C12

1 Introduction

For standard terminology and notation in graph theory we refer Harary³ and Zaslavsky¹⁶ for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

A great deal of work in twentieth century sociology has concerned itself with the behavior of small groups of individuals. Perhaps the simplest approach to studying such a group is to draw a digraph in which the individuals are the vertices, and in which there is an arc from vertex x to vertex y if x is in some relation to y , for example, if x likes y , x associates with y , x chooses y for a business partner, etc. Such a digraph is sometimes called a sociogram. Many of the relationships of interest have natural opposites, for example likes/dislikes, associates with/avoids, and so on. In that case, we can include two different relations in one digraph by using two different kinds of arc, or by using signs to distinguish them. Then, the presence of an arc

means that one of the relationships is present, and the + indicates one of the relationships, the positive one, while the – indicates the other relationship. For example, we might let an arc from x to y mean that x has strong feelings toward y , and put a + if these feelings are liking, a – if they are disliking. We obtain a signed digraph.

In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs + and – are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs + and – are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. Formally, a signed graph $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function that assigns a sign $\sigma(e) \in \{+, -\}$, to each edge in Γ . σ is called the signature or sign function. In such a signed graph, a subset A of $E(\Gamma)$ is said to be positive if it contains an even number of negative edges, otherwise is said to be negative. Balance or imbalance is the fundamental property of a signed graph. A signed graph Σ is balanced if each cycle of Σ is positive. Otherwise it is unbalanced.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an iso-morphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider⁶ who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary² provided a mathematical model for balance through graphs.

A marking of Σ is a function $\zeta : V(\Gamma) \rightarrow \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ .

The following are the fundamental results about balance, the second being a more advanced form of the first. **Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.**

Theorem 1. A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:

- (i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2 , and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary⁴).*
- (ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar⁸).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\bar{\Sigma} = (\bar{\Gamma}, \sigma')$, where for any edge $e = uv \in \bar{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\bar{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1. For more new notions on signed graphs refer the papers (see [10–13]).

A switching function for Σ is a function $\zeta : V \rightarrow \{+, -\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w . The switched signed graph is $\Sigma^\zeta := (\Sigma|\sigma^\zeta)$. We say that Σ switched by ζ . Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched signed graph is Σ^X . This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying

graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2 . The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see¹⁴) or *cycle isomorphic* (see¹⁵) if there exists an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2 . The following result is well known (see¹⁵):

Theorem 2. (T. Zaslavsky¹⁵) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In⁹, the authors introduced the switching and cycle isomorphism for signed digraphs.

In this paper, we initiate a study of the *super radial signed graph* of a given signed graph and solve some important signed graph equations and equivalences involving it. Further, we obtained the structural characterization of super radial signed graphs.

2 Super Radial Signed Graph of a Signed Graph :

In a graph Γ , the distance $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u . The radius $r(\Gamma)$ of Γ is defined by $r(\Gamma) = \min\{e(u) : u \in \Gamma\}$ and the diameter $d(\Gamma)$ of Γ is defined by $d(\Gamma) = \max\{e(u) : u \in \Gamma\}$. A graph for which $r(\Gamma) = d(\Gamma)$ is called a *selfcentered graph* of radius $r(\Gamma)$. A vertex v is called an eccentric vertex of a vertex u if $d(u, v) = e(u)$. A vertex v of Γ is called an eccentric vertex of Γ if it is an eccentric vertex of some vertex of Γ . Let S_i denote the subset of vertices of Γ whose eccentricity is equal to i .

In⁷, the authors introduced a new type of graph called super radial graph. The super-radial graph $SR(\Gamma)$ of a graph Γ on the same vertex set of Γ and two vertices u and v are adjacent in $SR(\Gamma)$ if and only if the distance between them is greater than or equal to $d(\Gamma) - r(\Gamma) + 1$. If Γ is disconnected, then two vertices are adjacent in $SR(\Gamma)$ if they belong to different components of Γ .

Motivated by the existing definition of complement of a signed graph, we now extend the notion of super radial graphs to signed graphs as follows: The *super radial signed graph* $SR(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $SR(\Gamma)$ and sign of any edge uv is $SR(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ . Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called super radial signed graph, if $\Sigma \cong SR(\Sigma')$ for some signed graph Σ' . The following result restricts the class of super radial graphs.

Theorem 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its super radial signed graph $SR(\Sigma)$ is balanced.*

Proof. Since sign of any edge $e = uv$ in $SR(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ , by Theorem 1, $SR(\Sigma)$ is balanced. \square

For any positive integer k , the k^{th} iterated super radial signed graph, $SR^k(\Sigma)$ of Σ is defined as follows:

$$SR^0(\Sigma) = \Sigma, SR^k(\Sigma) = SR(SR^{k-1}(\Sigma))$$

Corollary 4. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k , $SR^k(\Sigma)$ is balanced.*

The following result characterize signed graphs which are super radial signed graphs.

Theorem 5. *A signed graph $\Sigma = (\Gamma, \sigma)$ is a super radial signed graph if, and only if, Σ is balanced signed graph and its underlying graph Γ is a super radial graph.*

Proof. Suppose that Σ is balanced and Γ is a super radial graph. Then there exists a graph Γ' such

that $SR(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ' , $\sigma'(e)$ is the marking of the corresponding vertex in Γ . Then clearly, $SR(\Sigma') \cong \Sigma$. Hence Σ is a super radial signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a super radial signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $SR(\Sigma) \cong \Sigma$. Hence, Γ is the super radial graph of Γ' and by Theorem 3, Σ is balanced.

In⁷, the authors characterize the graphs for which $SR(\Gamma) \cong \bar{\Gamma}$.

Theorem 6. Let Γ be a graph of order n . Then $SR(\Gamma) \cong \bar{\Gamma}$ if, and only if, Γ is a graph with $d(\Gamma) = r(\Gamma) + 1$ or Γ is disconnected in which each component is complete.

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $SR(\Sigma) \sim \bar{\Sigma}$.

Theorem 7. For any signed graph $\Sigma = (\Gamma, \sigma)$, $SR(\Sigma) \sim \bar{\Sigma}$ if, and only if, Γ is a graph with $d(\Gamma) = r(\Gamma) + 1$ or Γ is disconnected in which each component is complete.

Proof. Suppose that $SR(\Sigma) \sim \bar{\Sigma}$. Then clearly, $SR(\Gamma) \cong \bar{\Gamma}$. Hence by Theorem 6, Γ is a graph with $d(\Gamma) = r(\Gamma) + 1$ or Γ is disconnected in which each component is complete.

Conversely, suppose that Σ is a signed graph whose underlying graph Γ is a graph with $d(\Gamma) = r(\Gamma) + 1$ or Γ is disconnected in which each component is complete. Then by Theorem 6, $SR(\Gamma) \cong \bar{\Gamma}$. Since for any signed graph Σ , both $SR(\Sigma)$ and $\bar{\Sigma}$ are balanced, the result follows by Theorem 2. \square

Let F_{11} and F_{22} denote the set of all connected graphs Γ for which $r(\Gamma) = d(\Gamma) = 1$ and $r(\Gamma) = 2$, $d(\Gamma) = 3$ respectively.

The following result characterizes the signed graphs which are isomorphic to super radial signed graphs. In case of graphs the following result is due to Kathiresan *et al.*⁷:

Theorem 8. For any graph Γ , $SR(\Gamma) \cong \Gamma$ if, and only if, either $\Gamma \in F_{11}$ or $\Gamma \in F_{22}$ with $\Gamma \cong \bar{\Gamma}$.

Theorem 9. For any signed graph $\Sigma = (\Gamma, \sigma)$, $\Sigma \sim SR(\Sigma)$ if, and only if, Σ is balanced and the underlying graph Γ belongs to either F_{11} or F_{22} with Γ is self-complementary.

Proof. Suppose $SR(\Sigma) \sim \Sigma$. This implies, $SR(\Gamma) \cong \Gamma$ and hence by Theorem 8, we see that the graph Γ satisfies the conditions in Theorem 8. Now, if Σ is any signed graph with underlying graph Γ belongs to either F_{11} or F_{22} with Γ is self-complementary, Theorem 3 implies that $SR(\Sigma)$ is balanced and hence if Σ is unbalanced and its super radial signed graph $SR(\Sigma)$ being balanced can not be switching equivalent to Σ in accordance with Theorem 2. Therefore, Σ must be balanced.

Conversely, suppose that Σ balanced signed graph with the underlying graph Γ belongs to either F_{11} or F_{22} with Γ is self-complementary. Then, since $SR(\Sigma)$ is balanced as per Theorem 3 and since $SR(\Gamma) \cong \Gamma$ by Theorem 8, the result follows from Theorem 2 again. \square

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in⁵ as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ . However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(\cdot)$ of taking the negation of Σ .

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $SR(\Sigma)$ is balanced (Theorem 3). We now examine, the conditions under which negation $\eta(\Sigma)$ of $SR(\Sigma)$ is balanced.

Theorem 10. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $SR(\Gamma)$ is bipartite then $\eta(SR(\Sigma))$ is balanced.

Proof. Since, by Theorem 3, $SR(\Sigma)$ is balanced, if each cycle C in $SR(\Sigma)$ contains even number of negative edges. Also, since $SR(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $SR(\Sigma)$ is also even. Hence $\eta(SR(\Sigma))$ is balanced. \square

Scope of Future Work :

As mentioned, one of the observations made by Kathiresan *et al.*⁷ is that $SR(\Gamma) \cong \overline{\Gamma}$ and $\Gamma \cong SR(\Gamma)$. However, this leads to characterize signed graphs Σ such that:

- $SR(\Sigma) \cong \overline{\Sigma}$
- $\Sigma \cong SR(\Sigma)$.

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