Ricci Solitons on three Dimensional β-Kenmotsu Manifolds with Respect to Schouten-van Kampen Connection

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Abstract

The object of the present paper is to study 3-dimensional β- Kenmotsu manifolds whose metric is Ricci soliton with respect to Schouten- van Kampen connection. We found the condition for the Ricci soliton structure to be invariant under Schouten-van Kampen connection. We have also showed that the Ricci soliton structure with respect to usual Levi-Civita connection transforms to a η-Ricci soliton structure under D-homothetic deformation. Finally we have shown that if a 3-dimensional β-Kenmotsu manifold admits a Ricci soliton structure with respect to Schouten-van Kampen connection and potential vector field as the Reeb vector field, then the manifold becomes K-contact Einstein.

Key words and phrases: Ricci soliton, *β*-Kenmotsu manifold, Schouten-van Kampen con- nection, *D*-homothetic deformation.

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1. Introduction

In 1982, Hamilton[12] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for

those manifolds with positive curvature. Perelman[27,28] used Ricci flow and its surgery to prove the Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphisms and scaling. To be precise, a Ricci soliton on a Riemannian manifold (M, g) is a triple (g, V, λ) satisfying

(1.1) $£_V g + 2S + 2\lambda g = 0,$

where S is the Ricci tensor, $£_V$ is the Lie derivative along the vector field V on M and $\lambda \in R$ [13]. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If λ is a smooth function on M then the metric satisfying (1.1) is called Ricci almost soliton[29]. In this connection it is mentioned that Hui and Patra[18] recently studied Ricci almost solition on Riemannian manifolds.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In[33] Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi *et al*.[1-3], Bejan and Crasmareanu[4], Blaga[6], Calin and Crasmareanu[8], Chen and Deshmukh[10], Deshmukh *et al*.[11], Hui *et al*.[9,14-17,19-21], Nagaraja and Premalatha[24], Tripathi[35] and many others.

In 1972, Kenmotsu[23] introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds. It is well known that odd dimensional spheres admit Sasakian structures whereas odd dimensional hyperbolic spaces can not admit Sasakian structure, but have so-called Kenmotsu structure. Kenmotsu manifolds are normal (non-contact) almost contact Riemannian manifolds. Kenmotsu[23] investigated fundamental properties on local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kahler fiber. As a generalization of both Sasakin and Kenmotsu manifolds, Oubiña[26] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kahler manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are respectively called the cosympletic, α-Sasakian and β-Kenmotsu manifold, α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively. As β is a scalar function, β-Kenmotsu manifolds provide a large varieties of Kenmotsu manifolds. β-Kenmotsu manifolds have been studied by several authors. In this connection it may be mentioned that Shaikh and Hui studied locally ϕ-symmetric β-kenmotsu manifolds[31] and extended generalized ϕ-recurrent β-Kenmotsu Manifolds[32], respectively. Also Calin and Crasmareanu[8] studied f-Kenmotsu manifolds.

The Schouten-van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection[5,22,30]. Olszak has studied Schouten- van Kampen connection adapted to an almost contact metric structure[25]. In[36], Yildiz *et al*. studied 3-dimensional f-Kenmotsu manifolds with respect to Schouten-van Kampen connection and obtained some interesting result.

Motivated by these, the present paper deals with the study of Ricci soliton on 3-dimensional β-Kenmotsu manifold with respect to Schouten-van Kampen connection. The paper is organized as follows. Section 2 is concerned with preliminaries. In section 3 we have studied D-homothetic deformation over a Ricci soliton structure. Finally in section 4 we have studied Ricci soliton on 3-dimensional β-Kenmotsu manifolds with Schouten-van Kampen connection.

2. Preliminaries :

A $(2n + 1)$-dimensional smooth manifold M is said to be an almost contact metric manifold[7] if it admits an (1,1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g, which satisfy

(2.1)   (a) $\phi\xi = 0$,   (b) $\eta(\phi X) = 0$,   (c) $\phi^2 X = -X + \eta(X)\xi$,

(2.2)   (a) $g(\phi X, Y) = -g(X, \phi Y)$,   (b) $\eta(X) = g(X, \xi)$,   (c) $\eta(\xi) = 1$,

(2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold M^{2n+1} (ϕ, ξ, η, g) is said to be a β-Kenmotsu manifold if the following conditions hold[23]:

(2.4) $\nabla_X \xi = \beta[X - \eta(X)\xi]$,

(2.5) $(\nabla_X \phi)(Y) = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X]$.

If $\beta = 1$, then a β-Kenmotsu manifold is called a Kenmotsu manifold; and if β is constant, then it is called a homothetic Kenmotsu manifold. In a β-Kenmotsu manifold, the following relations hold[23,26]:

(2.6) $(\nabla_X \eta)(Y) = \beta[g(X, Y) - \eta(X)\eta(Y)]$,

(2.7) $R(X, Y)\xi = -\beta^2[\eta(Y)X - \eta(X)Y]$
                           $+ (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\}$,

(2.8) $R(\xi, X)Y = [\beta^2 + (\xi\beta)][\eta(Y)X - g(X, Y)\xi]$,

(2.9) $\eta(R(X, Y)Z) = \beta^2[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]$
                           $- (X\beta)\{g(Y, Z) - \eta(Y)\eta(Z)\}$
                           $+ (Y\beta)\{g(X, Z) - \eta(Z)\eta(X)\}$,

(2.10) $S(X, \xi) = -\{2n\beta^2 + (\xi\beta)\}\eta(X) - (2n - 1)(X\beta)$

for all $X, Y, Z \in \chi(M)$.

Let M (ϕ, ξ, η, g) be an almost contact metric manifold. Then we have two naturally defined distribution in the tangent bundle TM of M as follows

$H = ker(\eta), \; V = span(\xi)$.

Then we have $H \oplus V = TM$, $H \cap V = 0$ and $H \perp V$. This decomposition allows one to define the Schouten-van Kampen connection $\widetilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\widetilde{\nabla}$ on a β-Kenmotsu manifold with respect to Levi-Civita connection ∇ is defined by[36]

(2.11) $\widetilde{\nabla}_X Y = \nabla_X Y + \beta[g(X, Y)\xi - \eta(Y)X]$.

If R and $\widetilde{R}$, S and $\widetilde{S}$ and r and $\widetilde{r}$ be the Riemann curvature tensor, Riccic urvature and scalar curvature in a 3-dimensional β-Kenmotsu manifold with respect to ∇ and $\widetilde{\nabla}$ respectively, then we have[36]

(2.12) $\widetilde{R}(X, Y)Z = R(X, Y)Z + \beta^2\{g(Y, Z)X - g(X, Z)Y\}$
                           $+ \dot{\beta}\{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi$
                           $+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$.

(2.13) $\widetilde{S}(X, Y) = S(X, Y) + (2\beta^2 + \dot{\beta})g(Y, Z) + \dot{\beta}\eta(Y)\eta(Z)$.

and

(2.14) $\widetilde{r} = r + 6\beta^2 + 4\dot{\beta}$.

for all $X, Y, Z \in \chi(M)$, where we are assuming $\dot{\beta} = \xi\beta$. A β-Kenmotsu manifold is said to be regular if

$\beta^2 + \dot{\beta} \neq 0$. The interesting fact about the connection $\widetilde{\nabla}$ is that the g, ξ and η are all parallel with respect to this connection.

3. D-homothetic deformation and Ricci soliton :

Let $M(\phi, \xi, \eta, g)$ be an almost contact metric manifold with $dim\ M = m = 2n + 1$. Then $\eta = 0$ defines a $(m - 1)$-dimensional distribution D on M. By an D-homothetic deformation[34], we mean a change of structure tensor of the form

(3.1) $\widetilde{\phi} = \phi,\ \ \widetilde{\xi} = \dfrac{1}{a}\xi,\ \ \overline{\eta} = a\,\eta,\ \ \ \text{and } \overline{g} = ag + a(a-1)\eta \otimes \eta,$

where a is a non-zero positive constant. This is to be noted that if $M(\phi, \xi, \eta, g)$ is an almost contact metric manifold then $\overline{M}(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is also an almost contact metric manifold. Now we have from (2.4) that

(3.2) $(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$
$= 2\beta[g(X, Y) - \eta(X)\eta(Y)].$

Since the Lie derivative operator only depends on the smooth structure of the underlying manifold so we have

(3.3) $(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$
$= 2\beta[g(X, Y) - \eta(X)\eta(Y)]$
$= (\pounds_\xi \overline{g})(X, Y).$

Again from (1.1) and (3.2) we have

(3.4) $S(X, Y) = -(\beta + \lambda)g(X, Y) + \beta\eta(X)\eta(Y)$

Now applying D-homothetic deformation we have from (3.4) that

(3.5) $\overline{S}(X, Y) = -\dfrac{\lambda + \beta}{a}\overline{g}(X, Y) + \dfrac{(a-1)(\lambda + \beta) + 1}{a^2}\overline{\eta}(X)\overline{\eta}(Y).$

Next from (3.3) and (3.5) we have

$(\pounds_\xi \overline{g})(X, Y) + 2\overline{S}(X, Y) + \overline{\lambda}\overline{g}(X, Y) + \overline{\mu}\,\overline{\eta}(X)\overline{\eta}(Y) = 0$

for all vector field $X, Y, Z \in \chi(M)$ with $\overline{\lambda} = \frac{\lambda - \beta}{a}$ and $\overline{\mu} = \frac{a\beta - a\lambda - \beta}{a^2}$. This shows that the structure $(\overline{g}, \xi, \overline{\lambda}, \overline{\mu})$ is an η-Ricci soliton structure on $\overline{M}$. Hence we can state the following:

Theorem 3.1. If (g, ξ, λ) is a Ricci soliton structure on M, then the D-homothetic deformation transforms the Ricci soliton structure into an η-Ricci soliton structure.

4. The Schouten-van Kampen connection and Ricci almost soliton :

Let the metric g of a 3-dimensional β-Kenmotsu manifold be a Ricci almost soliton with respect to ∇. Now

(4.1) $(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$
$= g(\widetilde{\nabla}_X \xi - \beta\{\eta(X)\xi - \xi\}, Y)$
$+ g(X, \widetilde{\nabla}_Y \xi - \beta\{\eta(Y)\xi - \xi\})$
$= g(\widetilde{\nabla}_X \xi, Y) + g(X, \widetilde{\nabla}_Y \xi) - \beta g(\eta(X)\xi, Y)$
$= \beta\eta(Y) + \beta\eta(X) - \beta g(X, \eta(Y)\xi)$
$= (\widetilde{\pounds}_\xi g)(X, Y) + \beta(\eta(X) + \eta(Y)) - 2\beta\eta(X)\eta(Y).$

Now from (2.13) and (1.1) we have

(4.2) $\qquad (\tilde{\pounds}_{\xi} g)(X, Y) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y)$

$\qquad = (2\beta - 2\dot{\beta})\eta(X)\eta(Y) - \beta\{\eta(X) + \eta(Y)\} - 2\{2\beta^2 + \dot{\beta}\}g(X, Y).$

Now if (g, ξ, λ) be a Ricci almost soliton structure on M with respect to Levi- Civita connection then the Ricci almost soliton structure is preserved for the Schouten-van Kampen connection if and only if

(4.3) $\qquad (2\beta - 2\dot{\beta})\eta(X)\eta(Y) = \beta\{\eta(X) + \eta(Y)\} + 2\{2\beta^2 + \dot{\beta}\}g(X, Y)$

holds for arbitary $X, Y \in \chi(M)$.

So in particular putting $X = Y = \xi$ in (4.3), we have $\beta^2 + \dot{\beta} = 0$. This leads to the following:

        Theorem 4.1. Let (g, ξ, λ) be a Ricci almost soliton structure on non-regular β-Kenmotsu manifold M with respect to Levi-Civita connection then the Ricci almost soliton structure is preserved for the Schouten-van Kampen connection.

        Next let (g, ξ, λ) be a Ricci almost soliton on M with respect to Schouten-van Kampen connection. Then we have

(4.4) $\qquad (\tilde{\pounds}_{\xi} g)(X, Y) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) = 0.$

But from (2.11) we have

$\qquad (\tilde{\pounds}_{\xi} g)(X, Y) = 0,$

which shows that the Reeb vector field is Killing. Hence (4.4) implies that

$\qquad \tilde{S}(X, Y) = -\lambda g(X, Y).$

Thus we can state the following:

        Theorem 4.2. If (g, ξ, λ) is a Ricci soliton on a β-Kenmotsu manifold with respect to Schouten-van Kampen connection, then the manifold is K −contact Einstein with respect to Scouten-van Kampen connection.

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